Eulerian or Lagrangian: The importance of reference frames for remote sensing

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Motivation

Various current (e.g. S-MODE, SWOT) and future (e.g. Odysea, Butterfly, Harmony) NASA funded projects all plan to use orbital and sub-orbital scatterometer measurements to infer surface ocean currents, though there are significant questions about the interpretations of these measurements. Specifically, scatterometers measure the Doppler-shifted frequencies of short (on the order of meters) surface gravity waves, and surface currents are derived by considering the inverse problem: 'Which currents generate this Doppler shift?' In particular, are these derived currents Lagrangian, Eulerian, or some mix of the two? Why would this be important? Such questions are the aim of this poster.

Definitions

As we believe much of the confusion surrounding this topic stems from misinterpretations of the Eulerian and Lagrangian reference frames, we explicitly define them below.

- A quantity is **Eulerian** if it can be expressed in terms of spatial coordinates (x, y). Thus an Eulerian quantity describes evolution at a fixed point in space.
- A quantity is **Lagrangian** if it can be expressed in terms of particle labels (a, b). Thus a Lagrangian quantity describes evolution following a fixed particle.

The **Stokes Drift** is defined as the difference between the mean Lagrangian and mean Eulerian currents

$$\mathbf{U}_S(y) \equiv \overline{\mathbf{u}_L(b)} - \overline{\mathbf{u}_E(y)}\,, \tag{1}$$

where the overlines represent time averages over a period in each respective reference frame. One often neglected issue with this definition is the validity of taking the difference of averages between two quantities with different dependent variables and more importantly different definitions of averaging as the Lagrangian and Eulerian periods are not equal [2]. What does it mean to take the difference between a Lagrangian and Eulerian velocity? To avoid this confusion, we will instead use the term **wave-induced mean Lagrangian drift** to refer to the mean Lagrangian velocity of fluid particles over the Lagrangian wave period $\overline{\mathbf{u}_L(b)}$.

The particle trajectories in irrotational, incompressible and inviscid deep-water surface gravity waves are open, leading to a net drift in the direction of wave propagation.

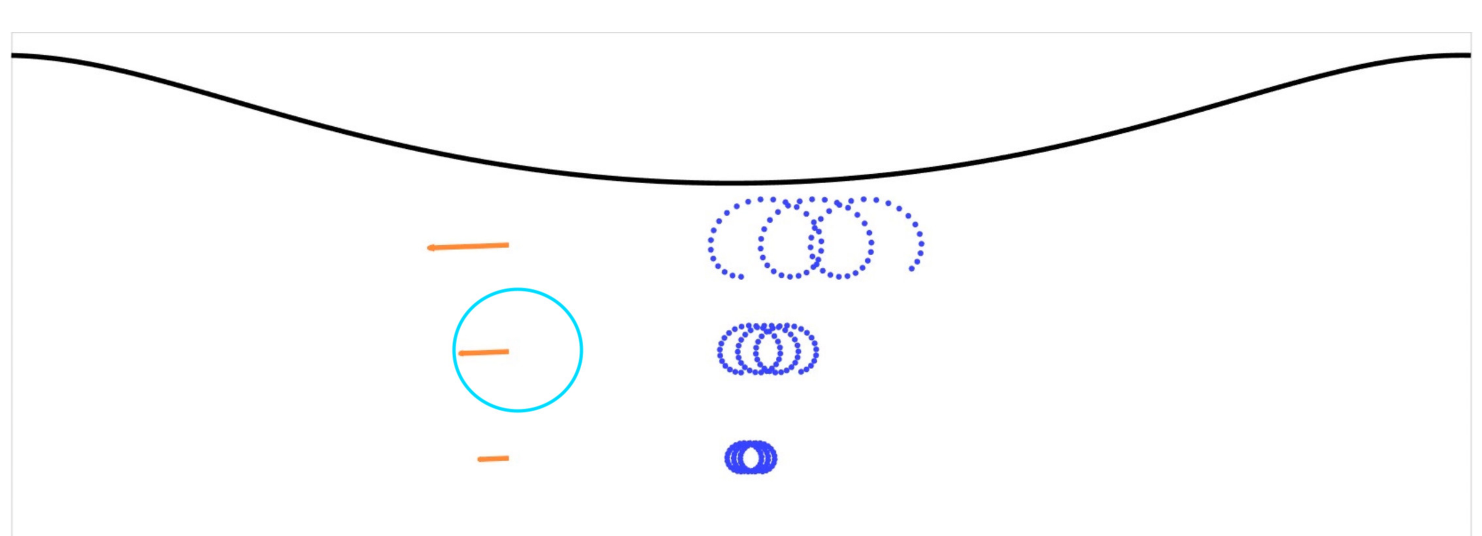


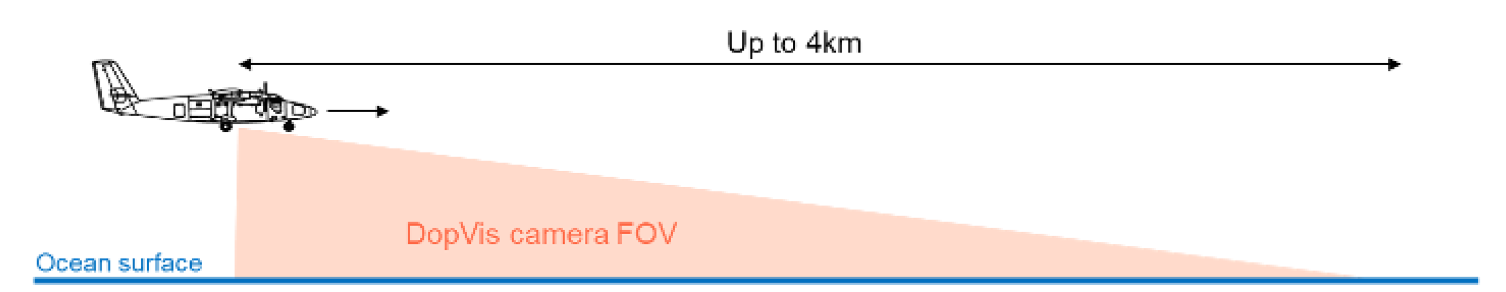
Figure 1. A schematic of the particle trajectories (blue) within an irrotational, incompressible, and inviscid surface gravity wave. A snapshot of Eulerian velocity vectors (orange), which exhibit periodic circular motion are shown at a fixed point in time.

Thus, at a fixed point in space, the average Eulerian velocity is zero, while for a fixed particle, the average Lagrangian velocity is nonzero. This difference is significant for ocean dynamics.

Why is this important?

As the Stokes Drift can be on the order of tens of centimeters per second, knowing whether or not this wave-induced mean Lagrangian motion is included in measured surface currents is crucial for platform measurements, dynamics, and surface advection. For some applications, such as determining the distribution of surface marine debris and plankton, a Lagrangian velocity is preferred. For a comparison with climate models, an Eulerian velocity is typically desired instead. The purpose of this poster is not to give preference to one reference frame or the other, but rather to highlight the importance of knowing in which reference frame a velocity is given.

MASS + DoppVis Instruments



One instrument heavily employed during the S-MODE campaigns was the MASS DoppVis instrument developed at the Air-Sea Interaction Lab (SIO). Using the DoppVis instrument, visible imagery of the ocean surface is collected using a camera pointing slightly ahead of aircraft, synchronized to a coupled GPS/IMU system. Each image is carefully georeferenced and combined into "3D video cubes" typically in the range of 128 to 512 meters in width, and 20 to 120 seconds in duration. Following the same approach described in [4], all cubes of space–time data are converted to wavenumber–frequency space using 3D fast Fourier transforms.

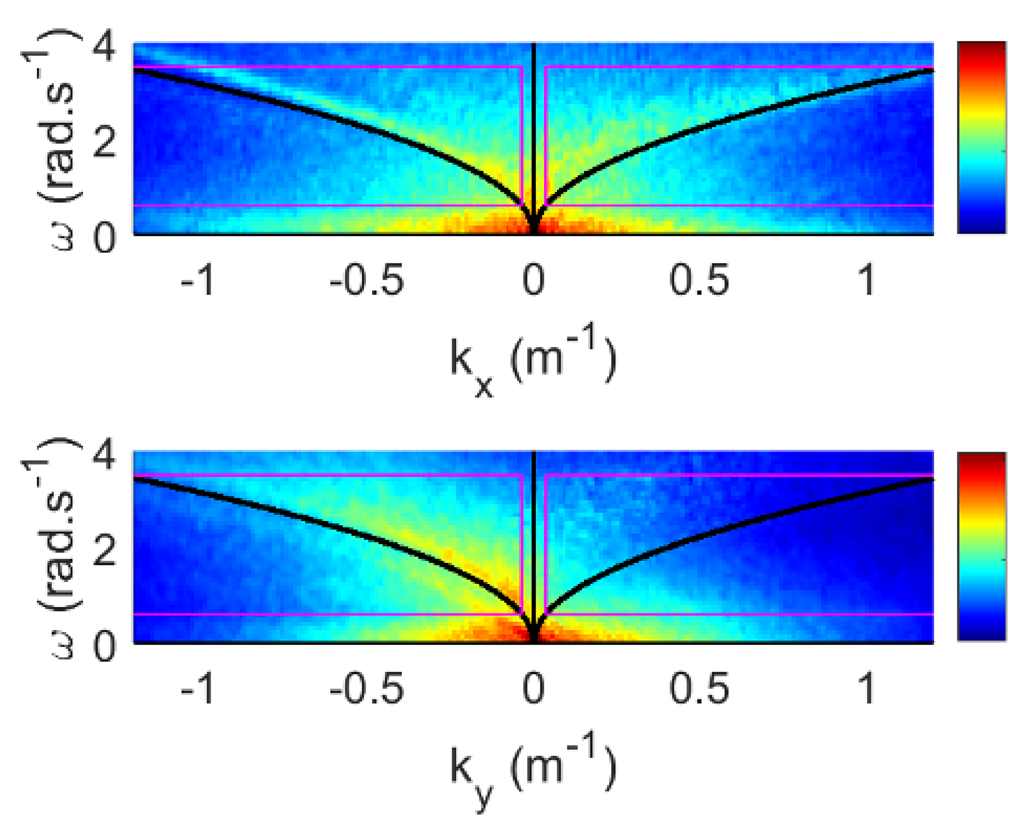


Figure 2. An example of a 3D spectra measured by DoppVis. The linear dispersion relation $\omega(k) = \sqrt{g|k|}$ is plotted in black.

Starting from the dispersion relation for surface gravity waves propagating on a depth-varying current, we have

$$\omega(\mathbf{k}) = \sqrt{gk} + \mathbf{c}(k) \cdot \mathbf{k}\,, \tag{2}$$

where $\mathbf{c}(\mathbf{k})$ is the Doppler shift to the phase velocities due to underlying currents, and k is the magnitude of the wavevector. By measuring the difference of the observed frequency (represented by the heatmap) and the linear dispersion relation (black line), we can determine $\mathbf{c}(k)$. The final step is determining how the doppler shift to the phase speed relates to the underlying currents.

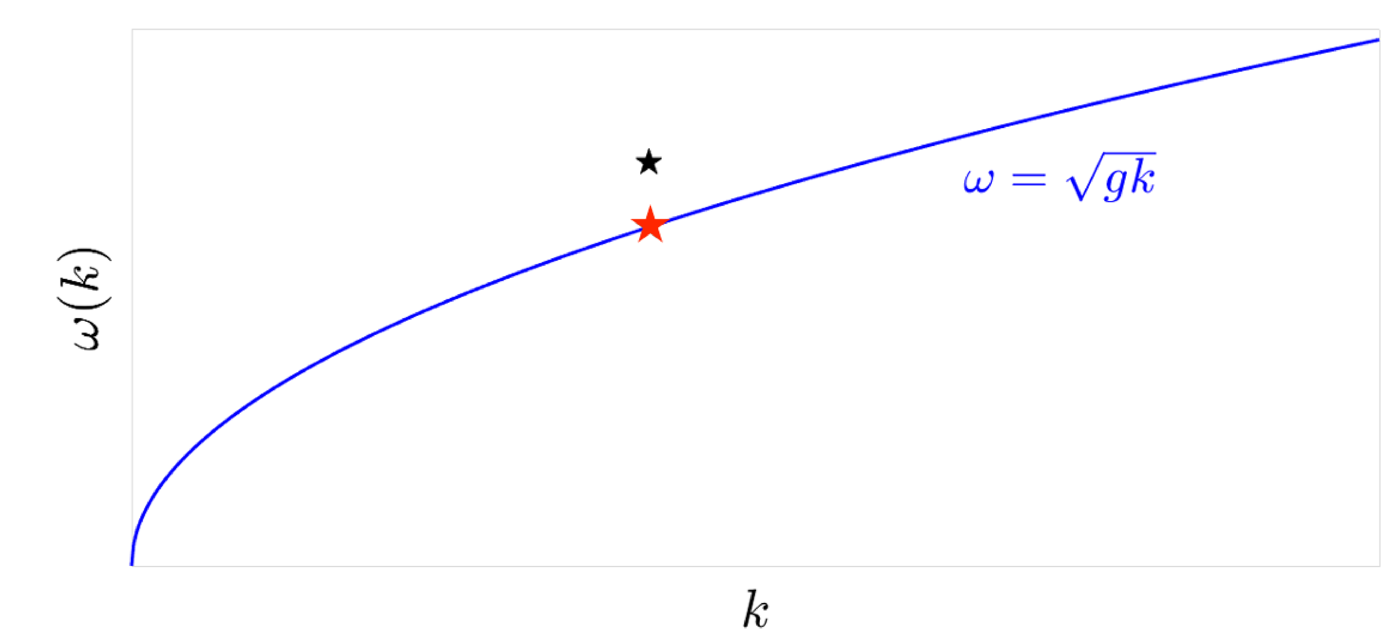
Phase speed modifications due to drift

Because all the particles are moving steadily with the mean Lagrangian drift as they complete their wavelike motion, the phase speed (which tracks the speed of the wave shape) must be increased. Because it is the particles themselves that undergo this drift, we must use *Lagrangian* quantities to determine this phase speed increase. From Abrashkin & Pelinovsky [1] and Pizzo et al. [3], the Doppler shift to the phase speed is a weighted sum of the mean Lagrangian drift

$$\mathbf{c}(k) = 2k\int_{-\infty}^{0} e^{2kb}\mathbf{u}_L(b)\,\mathrm{d}b\,, \tag{3}$$

necessarily small since the mean Lagrangian drift is small compared to the linear phase speed lest the waves overturn and break.

So which current does DoppVis measure?



Consider the image above as an idealized schematic of a potential 3D spectra for a single monochromatic wave (i.e. a delta function). Were the spike to occur at the red star, we would measure there to be no doppler shift, and hence there would be no inferred current. However, due to the presence of the mean Lagrangian drift, the phase speed is modified by equation (3) above. Therefore, we would actually observe the black star. Given that this is above the linear dispersion relationship (blue curve), we would measure a nonzero Doppler shift and infer a net current, which would be nothing but the mean Lagrangian drift $\mathbf{u}_L$. Therefore, **DoppVis and other similar instruments measure the mean Lagrangian currents, which include the wave-induced mean Lagrangian drift.**

Further questions:

- Which currents do wavegliders or saildrones measure?
- How easily can we convert these Lagrangian velocities to Eulerian ones? What are the associated challenges?

References

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[3] N.E. Pizzo, L. Lenain, O. Rømcke, S. Ellingsen, and B.K. Smeltzer. The role of lagrangian drift in the geometry, kinematics and dynamics of surface waves. *J. Fluid Mech.*, 954(R4), 2023.

[4] B. K. Smeltzer, E. Æsøy, A. Ådnøy, and S. Å. Ellingsen. An improved method for determining near-surface currents from wave dispersion measurements. *Journal of Geophysical Research: Oceans*, 124(12):8832–8851, 2019.