Physics of Fluids

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Cite as: Phys. Fluids **33**, 111706 (2021); https://doi.org/10.1063/5.0072166
Submitted: 20 September 2021 • Accepted: 24 October 2021 • Published Online: 15 November 2021

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ABSTRACT

We examine the role of wave packet bandwidth in modulating the breaking slope threshold for focusing deep-water surface gravity wave packets. Using a fully nonlinear potential flow solver and laboratory experiments, we show that the slope threshold may be strongly modulated by the wave packet bandwidth. We propose a new breaking threshold parameterization that shows that the slope threshold may be modulated by more than a factor of two by changes in the bandwidth. This has implications for parameterizations of the properties of the breaking induced flow (e.g., the energy dissipation) as they do not currently account for these effects.

Wave breaking at the ocean surface modulates the fluxes of mass, momentum, heat, and energy between the air and the sea.[1] Recently, there has been much progress made in parameterizing some of these effects based on scaling arguments involving the slope of the wave field and a parameter for the breaking threshold, taken to be a constant.[2–5] Using theoretical and numerical arguments, Pizzo and Melville[6] argued that the breaking threshold is not a constant and instead should also be a function of the wave packet bandwidth. However, that study was limited, leaving open the question of how significant the modulation to the wave slope might be. Here, we fill in this fundamental shortcoming by numerically examining a much broader region of phase space, finding that the breaking threshold may be significantly modified by the wave packet bandwidth and validate these results using data from archived laboratory experiments.[7,8]

Longuet-Higgins and Dommermuth[9] showed that very steep permanent progressive deep-water waves (here the wave slope ak was close to the limiting value of ≈ 0.443) subjected to normal mode perturbations may rapidly overturn and break because of the so-called superharmonic instability.[10–14] These waves contain a single dominant wavenumber (and frequency) as well as higher order (bound) harmonics. On the other hand, Dold and Peregrine[15] showed that a narrow-banded periodic wave packet with relatively small initial slope (around 0.1) may result in wave breaking. These studies hint at the role of wave packet bandwidth in setting a breaking slope threshold. The dependence of slope threshold on wave packet bandwidth is also implicit in the numerical work of Deike *et al.*,[16] where the authors found a larger breaking threshold than the corresponding (broadband) laboratory experiments. Additionally, the laboratory studies of Drazen *et al.* (Ref. 2, in particular see their Fig. 9) show a change in the strength of the breaking event (e.g., spilling vs plunging) for fixed wave slope and varying bandwidth, finding that increased bandwidth corresponded to a weaker type of breaker.

Considerable work on wave breaking criteria has been performed over the years and is discussed in the comprehensive review by Perlin *et al.*[17] These criteria generally fall into three categories: geometric, kinematic and dynamic. Ample work has been done on each of these criteria and recently there has been success in applying kinematic criteria to determine which waves may break.[18–21] However, these are local criteria and are sensitive to both the definitions of important physical quantities, such as the wave phase speed, as well as the location at which the criteria are evaluated, i.e., the highest part of the wave (i.e., the crest) vs the steepest part of the wave. Here we propose *predictive* global criteria that only depends on the initial conditions of these focusing wave packets, removing the need for characterizing many of the rapidly varying processes that occur during focusing.

Additionally, it has been shown in the laboratory[2,8] that a linear relationship exists between S, the linear prediction of the slope at focusing, and hk_c, where h is the height of the wave at breaking and k_c is the central wavenumber of the wave packet. This implies that parameterizations of the breaking induced flow, which use h as a length scale,[2,4,16,22–24] can be modeled using S, a variable that is known *a priori*. In that context, parameterizing the breaking threshold as a function of the variables characterizing the initial wave packet, as opposed to local variables at breaking,[18,19] becomes particularly relevant.

Recently, the role of wave packet bandwidth was examined in a laboratory study on the mass transport[4,5,7] and energy dissipation induced by deep-water wave breaking.[8] The authors found that the bandwidth can strongly modify the breaking induced transport. Furthermore, they found that the energy dissipation was further collapsed by considering bandwidth effects (see their Fig. 8). This highlights the need to better understand the role of this parameter in modulating properties of the breaking induced flow.

The outline of this paper is as follows. We first review properties characterizing focusing deep-water wave packets. Next, we discuss a comprehensive set of numerical experiments. Archived data on the breaking threshold from laboratory experiments is then presented. Finally, the results are discussed.

When wind blows over deep-water, a finite bandwidth of waves is created.[25] These waves are dispersive, and as such may constructively and destructively interfere, leading to wave breaking. To reproduce this behavior in the laboratory, Longuet-Higgins[26] proposed generating wave packets of the form

$$\eta = \sum_{n}^{N} a_n \cos\theta_n, \tag{1}$$

where η is the free surface displacement, a_n are the wave amplitudes and $\theta_n = k_n(x - x_b) - \omega_n(t - t_b)$. Here $\omega_n^2 = gk_n \tanh k_n H$ for ω_n the angular frequency, k_n the associated wave number, H the depth of the fluid and N the number of components included in the wave packet. Additionally, x_b and t_b are the linear prediction of the breaking locations and times, respectively, representing the point of maximum focusing according to linear theory. There are two main parameters that characterize these wave packets. The first is the linear prediction of the maximum modulus slope at focusing, defined as $S \equiv \sum a_n k_n$. Additionally, the wave packets are defined over a finite band of frequencies/wavenumbers so that $\omega_n = \omega_0(1 - \Delta\frac{n-N/2}{N})$. Note, this formula only applies when $n > 0$.

We define Δ as the non-dimensional bandwidth. The modes within the wave packet, which collectively define the bandwidth, are grouped around a chosen central frequency f_c. This central frequency is used to define ω_n, k_n, a_n, and a windowing function, which is then applied to the wave packet so that only a single compact wave packet is considered.[8,27]

Physically, the linear prediction of the slope S is a measure of the nonlinearity of the system, while the bandwidth Δ sets the time (or equivalently space) scale over which the waves interact. These two parameters then set the maximum slope at focusing. The central goal of this manuscript is to propose a model describing whether or not a wave will break as a function of its bandwidth and the linear prediction of the slope at focusing.

Using a series of theoretical and numerical arguments based on a higher order nonlinear Schrödinger equation, Pizzo and Melville[6] proposed that for narrow-banded deep-water waves, the slope at focusing is a function of Δ and S. Physically, there is a competition between linear dispersion, which acts to spread out action density, and nonlinearity, which can enhance focusing. Pizzo and Melville[6] elucidated the role of various terms in the higher order Schrödinger equation and showed that the asymmetric self-steepening term is significant in generating larger slopes.[28] Note, this term is also responsible for increasing the action density centroid velocity of nonlinear wave packets.[23]

To examine the breaking threshold dependence on bandwidth, we numerically integrate the equations of motion for fully nonlinear irrotational inviscid deep-water surface gravity waves. The numerical scheme of Dold and Peregrine[15] is used (see also Dold[29]) to study the focusing wave packets discussed above. Briefly, the model solves Laplace's equation in the interior of the fluid together with the boundary conditions for unforced irrotational surface gravity waves using Cauchy's integral theorem. The model uses a mixed Eulerian–Lagrangian approach, exploiting the advantages of working from a velocity potential, i.e., only needing to solve for the flow along the surface to completely determine the flow throughout the fluid, while using the position of Lagrangian particles along the free surface as dependent variables. This parametrization of the free surface in terms of Lagrangian particles allows the numerical scheme to integrate past the point where the waves overturn to examine phenomena occurring during breaking, but before the free surface has reconnected (see also Ref. 30).

We employ this numerical scheme to examine a range of initial bandwidths and slopes for the initial conditions defined by (1). The central wavenumber and central frequency of the waves are fixed at $(2\pi)^2/\mathrm{g\,m^{-1}}$ and 1 Hz, respectively, and we set the length of the channel to have a physical value of 100 m. The water is taken to be infinitely deep. The distance to breaking x_b depends on the bandwidth Δ such that $x_b = 10/\Delta$. We take $N = 100$ here, while the wave spectrum is defined by the bandwidth and the central frequency. Additionally, following Drazen *et al.*[2] we fix the slope of each component of the initial wave packet, i.e., $a_n k_n = S/N$, which sets the amplitude of the components.

We define a wave as having broken when its free surface becomes multi-valued. The resolution of our simulations is increased until we find that the breaking results have converged (specifically we have 2048 points in our domain). More specifically, we computed the L_2 norm of the free surface displacement as a function of resolution and ensured that the results converged to sufficient accuracy. Additionally, although the results presented are based on simulations using 2048 points, we also performed several integrations for points near the breaking threshold using 4096 points and found identical results. Note, as discussed in Longuet-Higgins and Cokelet,[30] the free surface points are Lagrangian and tend to cluster in regions of high curvature, leading to enhanced resolution near breaking. An example of a breaking wave from these simulations is shown in Fig. 1. In total, we performed numerical simulations for 218 cases.

Figure 2 shows the breaking slope threshold as a function of bandwidth computed from these numerical simulations. The blue region shows the waves that have broken, while the gray region shows the waves that have not broken. The white circles show the points in phase space that we examined numerically. We clearly see that the

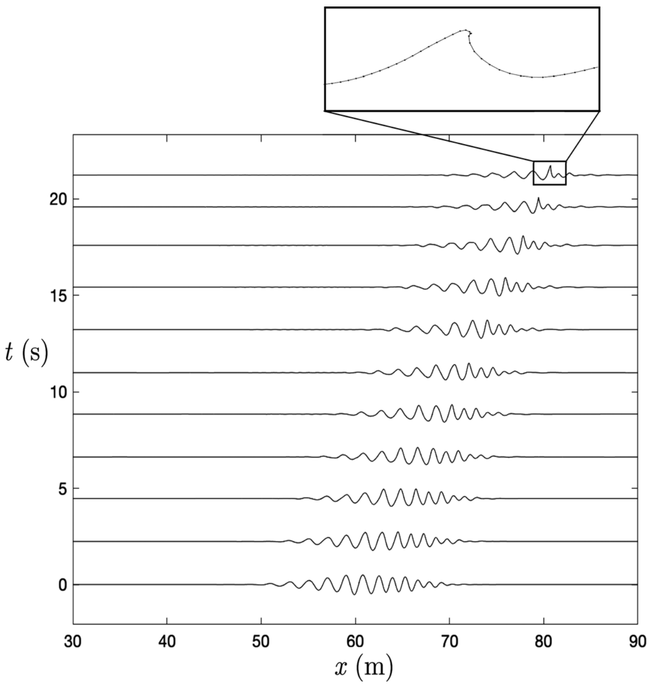

FIG. 1. The free surface displacement from numerical integrations of the fully nonlinear potential flow equations for deep-water free surface gravity waves. This focusing wave packet is constructed by generating longer faster waves after shorter slower waves, leading to a localization of energy density and subsequently wave breaking, shown in the inset. Here, the packet parameters are such that the linear prediction of the slope at focusing $S = 0.31$ while the wave packet bandwidth is 0.3. The black dots in the inset show the fluid particle locations, which are the dependent variables of the numerical scheme, at breaking. The goal of this paper is to better understand how wave packet bandwidth impacts whether or not wave breaking will occur.

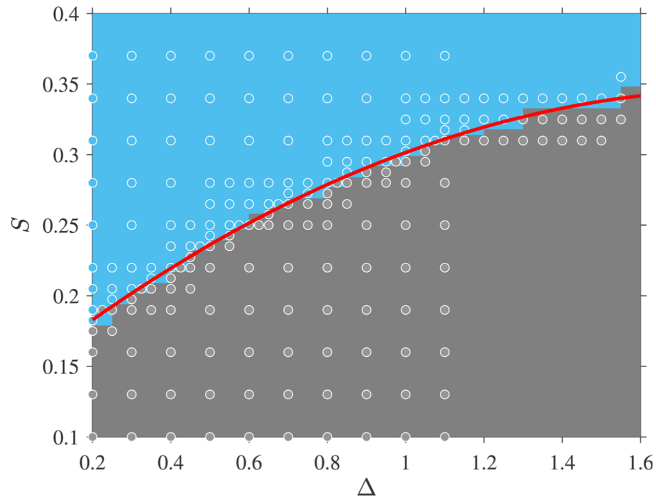

FIG. 2. Using the numerical scheme described in the body of the text, we investigate the role of the linear prediction of the slope at focusing, S, and wave packet bandwidth, Δ, on the wave breaking threshold, S_*. We display a phase space plot of breaking (shown in blue) vs non-breaking (shown in gray) as a function of Δ and S. The best fit is $S_* = -0.0579\Delta^2 + 0.2177\Delta + 0.1417$ for $0.2 < \Delta < 1.6$, with $R^2 = 0.99$. The results found here are in agreement with the theoretical analysis of Pizzo and Melville,[6] which predicted that waves were more likely to break for larger slopes and smaller bandwidths. The quantitative form of the line partitioning non-breaking and breaking waves was also predicted in that study using a simplified toy model based on a higher order nonlinear Schrödinger equation.

breaking threshold increases with bandwidth, with a quadratic relationship between the slope threshold S_* and the bandwidth, which takes the form

$$S_* = -0.0579\Delta^2 + 0.2177\Delta + 0.1417, \tag{2}$$

over the range of Δ considered here. This fit, shown in red in Fig. 2, does a remarkable job of parameterizing the partitioning of phase space between non-breaking and breaking waves.

This empirical fit is consistent with the theoretical and numerical study conducted by Pizzo and Melville.[6] There, they examined a higher order nonlinear Schrödinger equation. Specifically, to analytically examine focusing wave packets, they considered a Ritz optimization computation of a chirped Gaussian wave packet and examined the evolution of physically significant moments of the action distribution (see also Ref. 23). Using this technique, which also made use of the variational structure of the equations of motion derived using Whitham's method,[31,32] the author's found that there was a quasi-self-similarity to wave packet focusing governed by this equation as function of S and Δ. The authors found a complex relationship between the maximum slope at focusing and the initial conditions, but showed that it could be well approximated by a quadratic relationship between S and Δ, which is consistent with Eq. (2). Note, the analysis of Pizzo and Melville[6] was necessarily complicated, as the mathematical analysis of the higher order Schrödinger equation is algebraically complex. However, the physical ideas discussed qualitatively below, are relatively straightforward and for clarity of presentation and in order to not distract the reader from our relatively simple result, the full details of that analysis are not presented here. We direct the interested reader to that paper.

Physically, the arguments presented in Pizzo and Melville[6] quantify and qualify the role of the (finite bandwidth) four wave resonance in steepening surface waves vs the tendency of dispersion to spread out wave action. The balance of these terms, which is reminiscent of a simpler wave breaking model in shallow water,[33] dictates whether or not the packet will steepen enough so that the free surface becomes multi-valued. Note, the complex nature of the Schrödinger equation makes analysis significantly more difficult than the shallow water (modified) KdV equation. Unless strong ansatz are made (e.g., relating the wave phase to its amplitude), it is not obvious how to generate a simple coarse model of the breaking process in deep-water (see Pizzo and Melville[23] and Pizzo and Melville[6] for more details). Note, this model does not include higher-order resonances such as the superharmonic instability which leads to rapid overturning and breaking which partially motivated the authors to numerically examine these results.

Sinnis *et al.*[8] and Lenain *et al.*[7] conducted laboratory experiments at the Hydraulics Laboratory, Scripps Institution of Oceanography, to characterize mass transport induced by breaking deep-water focused wave packets.[4,5] As part of the project, the breaking threshold, defined here as $S_* = S$ when the free surface first entrains air, as detected by a side-looking camera, was carefully measured for the range of bandwidths Δ considered in those works.

Note, the laboratory experiments were necessarily conducted in finite depth (see Refs. 7 and 8 for further details). This modifies the focusing process, as the dispersive and nonlinear properties of the waves are modified in finite depth. In order to validate our numerical simulations against the laboratory cases, we computed the breaking threshold at the water depth used in the laboratory (in particular 0.5 m). This is shown in Fig. 3, where we find relatively good agreement between the numerical predictions and the observations.

One of the main results of this work is that the phase space is partitioned between non-breaking and breaking waves as a function of both bandwidth and slope. This partitioning takes the form of a quadratic relationship with the bandwidth, which is consistent with the theoretical arguments of Pizzo and Melville.[6]

Note, the laboratory experiments of Wu and Yao[34] also examined the breaking threshold as a function of bandwidth for focusing wave packets on a current. They find that the local slope *decreases* with increasing bandwidth. To the contrary, our model based on global properties of the wave field implies that the linear prediction of the local slope threshold increases with increasing bandwidth. It is unclear at this stage why these two studies find such contradictory results. A major challenge of the laboratory results is that they are constrained by the physical size limitation of their wave channel, that could limit their ability to properly characterize breaking threshold for small bandwidths in particular. The numerical experiments discussed here were conducted over a much larger distance, allowing the weak nonlinearity more time and space to act on the wave field. Note, Sinnis *et al.*[8] also find breaking threshold slope increasing with bandwidth in their laboratory experiments, consistent with the present study.

The results presented here have important implications for modeling properties of the breaking induced flow. In recent years there has been much progress made in proposing simple scaling arguments for the breaking induced flow based on variables characterizing the wave packets, including the energy dissipation rate,[2,3] the circulation induced by breaking,[22] the volume entrained by breaking,[35] and the mass transport due to breaking.[4] These models rely on a proper definition of the breaking threshold, as it is the difference between the wave slope and the threshold that dictates the magnitude, or strength, of the breaking event. Here, we have shown that the breaking threshold should vary as a function of the bandwidth, effectively modulating the strength of the breaking event, $S - S_*$. The wave field bandwidth can be estimated from spectral moments,[36] enabling simple implementation into coupled air-sea models that include wave breaking processes.

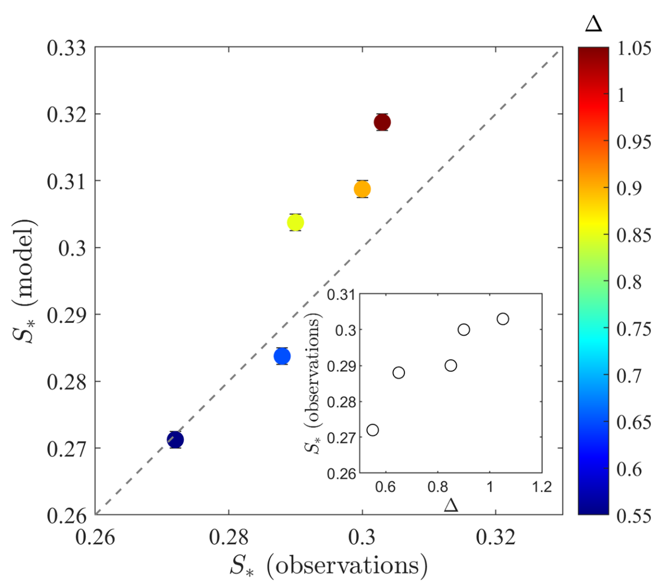

FIG. 3. Observed and predicted breaking threshold S_* for the laboratory experiments presented in Sinnis *et al.*[8] and Lenain *et al.*[7] These laboratory experiments used the same dispersive focusing employed in the numerical experiments to investigate properties of the breaking induced flow. The inset shows the breaking threshold S_* as a function of bandwidth, consistent with the trend found in Fig. 2.

The authors gratefully acknowledge numerous discussions with Ken Melville on this topic prior to his death in the autumn of 2019. Additionally, we thank two anonymous referees for helpful comments that have improved the manuscript.

This research was supported by grants from the Physical Oceanography programs at ONR (Grant Nos. N00014-17-1-2171, N00014-14-1-0710, N00014-17-1-3005, and N00014-19-1-2635), NSF (OCE; Grant No. OCE-1634289), and NASA (Grant No. 80NSSC19K1688).

AUTHOR DECLARATIONS

Conflict of Interest

The authors report no conflict of interest.

DATA AVAILABILITY

The data that support the findings of this study will be openly available in a UCSD repository that provides a DOI, Ref. 37.

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