Interaction of Water Waves and Currents

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I. Introduction

A. Summary

The varied physical circumstances in which interactions between water waves and currents occur are described in this introduction. Different mathematical approaches, relevant observations, and experiments that are applicable to all or some of these physical circumstances are described in the other sections. The paper has been written with gravity waves and currents such as those in seas or rivers in mind: thus there are only incidental references to the effects of surface tension and viscosity, which are of greatest significance for length scales of the order of centimeters and smaller. The emphasis is on waves and their interaction with preexisting currents rather than on wave-generated currents, although these are mentioned where they are relevant.

In all water wave problems approximations must be made to find mathematical solutions in order to gain physical understanding. Almost always the water is supposed to be inviscid and the flow irrotational. Here the first of these approximations is made but in few cases can the second approximation hold. Another common simplifying assumption is that the waves are of sufficiently small amplitude for the free surface boundary conditions to be linearized and evaluated at, or close to, the mean free surface. Most progress can be made in this subject with such a constraint, but wherever possible finite-amplitude effects are discussed. In order to get a reasonably wide class of solutions further approximations are necessary, the most important being for short waves and long waves, that is, for waves short (or long) compared with the length scale in which significant current variations occur. Sections II and III are on large- and small-scale currents, respectively.

Much of the theory of water waves on large-scale currents only differs in detail from theory for any short waves in a moving medium (e.g., see Bretherton, 1971, for a review of linear theory). An adequate theoretical description was first given by Longuet-Higgins and Stewart (1960, 1961), who introduced the idea of radiation stress. Further improvement in our understanding has come from the use of Whitham's method of averaging a Lagrangian and the concept of wave action. Since this is probably the most important field of wave–current interaction a number of simple situations are examined in detail in Section II.

Relatively little work has been done on small-scale currents, and Section III is mainly about waves in the presence of thin shear layers.

Unlike some other common forms of wave motion, water waves involve water motion varying with direction perpendicular to the space in which they propagate. That is, water waves propagate on the surface of the water, but their motion also varies with the depth. Thus if there is current variation with depth it may affect the waves on the water surface. This is the topic of Section IV, which includes appreciable detail because of the possible applications to waves on streams and to the effects of a wind-driven surface current.

In most applications the currents are turbulent and are approximated by a corresponding mean flow. However, there is interaction between waves and turbulence, so the few results are discussed in Section V.

The paper concludes with another short section on the interaction of waves generated by a ship with the flow around it.

A number of new results are incorporated in the text at various points. Particular examples are the errors involved in neglecting currents (Section II,B), the behavior of small-amplitude waves at a stopping point (Section II,D) and of finite-amplitude waves approaching a caustic (Section II,E), and the surface layer solution for waves above a critical layer Section IV,B).

B. Sea Waves

Over a large part of the world's oceans and seas the spatial distribution of surface currents due to the tides and ocean circulation is on such a large scale that even the largest ocean waves are on an effectively uniform current, unless global propagation is being considered. It is mainly near continental margins and in shallow seas that currents influence waves significantly and this is discussed further in Section I,C. However, the strong western boundary currents of oceans can and do strongly affect ocean waves. This is especially true of waves propagating onto and against such currents. They are shortened and steepened by the adverse current and refracted into caustics and foci by the shear at the currents' boundaries. Examples of damage to ships by waves on the Agulhas current are mentioned in Section II,E.

Even on a uniform current difficult problems of practical importance arise. This is particularly so with nonlinear properties of waves such as the forces they exert on structures. With the increasing number of offshore structures the prediction of forces due to combinations of currents and waves is of growing importance. An appreciation of the problem may be gained from Hogben (1974).

The surface drift caused by wind stress plays a role in wave dynamics as Banner and Phillips (1974) show (Section IV,C). It is probably also important in the generation of waves by wind; the tangential stress at the interface is influenced by its velocity and it in turn will influence the flow of air over the wave. Phillips and Banner (1974) have studied this boundary layer in the water, but its implications for the wind–wave system need further study.

Another effect, whose importance is debatable but not negligible, is the interaction between water waves. The interaction of short waves with much longer waves can be treated in the same manner as interaction with a current (Section II,F) and such effects are easily observed when a long swell meets short wind waves. Less commonly seen is the interaction of short waves with the velocity field of long internal waves (Section II,D) but Gargett and Hughes (1972) show photographs of wave patterns that are best explained in this manner.

The shortest waves are capillary waves, and their ubiquity among wind waves is in part due to the steep gravity waves. The small radius of curvature at the crest of the steepest gravity waves and the surface tension there act rather like a moving pressure distribution and generate capillary waves. However, in Section II,F another mechanism for forming such waves is pointed out. Short gravity–capillary waves being overtaken by a larger gravity wave can be reflected near its crest and propagate away from it as capillary–gravity waves. The wavelength of this second class of waves is longer than those generated by the first mechanism.

C. Coastal Waves

Water waves have their greatest economic importance when they arrive at coastlines. This is due to their ability to erode and build up land, their power to damage man-made structures, and the difficulties they cause in the handling of ships. Practically all the work to date on predicting and assessing coastal wave problems has neglected their interaction with currents, even though currents are often strong in coastal regions. There are two main reason for this neglect. One is that the transformation of waves due to lessening depth and their refraction and diffraction by underwater and coastal topography are often more important.

The other reason is that the currents are often inadequately known. This is because of their complexity. Tidal currents vary in cycles so that in many parts of the world a good measurement of them requires recording for at least a lunar month. Furthermore, local winds, especially in storms, can produce currents comparable in magnitude with those due to tides in many places. Man-made structures are often associated with river mouths (which may be the reason for a harbor's existence) and the river flow introduces further variability.

To complicate matters still further, the mass transport associated with an irrotational wave train usually sets up mean currents in the region of the shore. For more detail, Longuet-Higgins (1972) gives an account of the longshore currents generated by waves and James (1974) gives further detailed calculations. The waves also generate rip currents and other return flows, which complete a cycle of interaction by influencing the incoming waves (Section II,D). These effects are observed in hydraulic models, but sometimes efforts are made to minimize rather than measure them, especially if the wave-generated currents are unsteady.

It is not clear whether attempts should be made to incorporate current effects into most coastal wave studies. The substantial variations in typical current fields mean that getting the basic data is very expensive unless there is a strong, well-defined current system that clearly cannot be ignored. On the other hand, it is advisable to estimate the "worst" conditions together with their probability of occurrence. As is indicated in Section II,F, the common impression that waves are higher at high tide than at low tide receives support from theory, but focusing effects are probably more important. For example, tide races, in which steep waves occur on strong currents around headlands and in channels, are due to waves, propagating partially against the current, being concentrated. Fortunately such concentrations of wave energy are usually in the strongest current, which is not usually a point where structures are erected. Indeed, wave action at the shoreline may be substantially overestimated in some circumstances if currents are ignored, since they can have a sheltering effect.

Right on a beach, in the surf zone, there are appreciable problems in actually describing the waves. One promising model is to use the finite-amplitude shallow-water equations with bores fitted. In such a model a separation into waves and currents does not assist analysis. The relatively small scale of the currents also accentuates the difficulties. For example, rip currents might be expected to accentuate waves incident upon them; but observations indicate that waves on rip currents tend to be lower and break less. This is probably a case of diffraction being more important than refraction.

Two interesting minor points are a proposal by Dagan (1975) that short wave–long wave interaction may contribute to wave breaking (Section II,F), and an occasional wave feature in the backwash from surf that Peregrine (1974) interprets as being due to current shear in the vertical (Section IV,C).

D. Waves in Rivers and Channels

Rivers normally have a nonuniform current distribution and this directly affects all the waves that occur on rivers (with the possible exception of flood and tide waves). If the currents are sufficiently swift then variations in the river's bed and banks (e.g., a projecting obstacle) may cause stationary waves on the surface, like ship waves. If such waves are caused by a constriction in the channel they are usually confined to the region of maximum current velocity and a train of several crests may be seen (Section II,D). Stationary waves caused by a sluice or weir may have a very large amplitude (surface shear waves, Section IV,C).

Shorter waves are commonly generated by the wind and by boats. The most usual interaction of these waves with a typical current profile (i.e., maximum velocity away from the shore) is for waves traveling upstream to be refracted toward the maximum current, while waves traveling downstream are refracted toward the bank (Section II,E). Thus, in midstream the former waves may persist for a considerable time since they are not dissipated by interaction with the banks, while the latter waves soon meet the bank and decay rapidly. This also means that river banks may suffer more wave action from the upstream direction.

The variation of current along a river also affects waves (Section II,C). For example, a boat traveling upstream at a constant speed relative to the water produces waves of constant amplitude, but if they propagate upstream into a region of stronger current they may be considerably amplified. Thus, it is possible in such a region to see a boat pass traveling upstream and for the waves following it to increase continually in amplitude for a considerable time after the boat has passed. The author has experienced this while sculling and written a note about it (Peregrine, 1972). Similarly, the boat may generate waves that are "stopped" (that is, propagating upstream but with a group velocity equal to the stream velocity). These waves, which appear to be moving since their phase velocity is upstream, persist until dissipated, which may take a surprisingly long time.

E. Hydraulic Breakwaters

A hydraulic breakwater is simply an extensive current of water directed toward waves in order to stop them. It works; but for long waves very substantial currents are needed. The power needed to pump the water is such that it is rarely an economically feasible proposition. A considerable number of experiments have been performed—most designed to assess the power requirements of specific designs. Evans (1955) gives an historical perspective as well as some experimental results. More recent reports of experiments are Nece *et al.* (1968), Bulson (1963), and Williams and Wiegel (1962).

A closely related device is the bubble or pneumatic, breakwater. The original idea was that a stream of air flowing up through the water would form a region with a lower effective density than water, thus reflecting some of the wave energy. In fact, the entrainment of water by the air results in an outward surface current from a line of bubble generators, which is effective in stopping waves. Evans (1955) also includes and compares results from a pneumatic breakwater. A more recent paper is that by Green (1961), which also has a summary of previous work.

Naturally there is no need for a breakwater current to extend down to the full depth of the water if the incident waves are in "deep water." Experiments such as Evans' (1955) show that a surface current need only extend to a depth that is only a small fraction of a wavelength in order to stop waves, if it is strong enough. While there is linear and nonlinear theory available for waves on a slowly varying current that is uniform with depth (Section II,D), when the vertical structure of the current is also important results of linear theory are only sufficient to find the local wavelength, although this gives a first approximation to the stopping velocity (Taylor, 1955). Witham's method of using an averaged Lagrangian may provide a way of finding the variation of wave amplitude in such cases.

F. Ship Waves

The greatest interest in ship waves is in their contribution to resisting a ship's motion. This may be assessed by measuring the waves radiated by a ship. However, some of the energy and momentum that may be assigned to the wave field close to the ship is lost from the wave field, for example, by wave breaking. On the other hand, the waves generated by a ship depend on the flow of water around it, and if this is altered (for example, by boundary layer suction) then so is the radiated wave pattern. The interactions between flow and waves are discussed in more detail in Section VI.

Theoretical methods of estimating wave resistance are complementary to the measurements since these are usually made on ship models and need to be scaled for use in ship design. Most mathematical models assume inviscid irrotational flow, but it has been shown by experiment and theory that the wake and boundary layer lead to significant effects on the waves. These are not adequately described by increasing the size of the ship to account for the displacement thickness of the boundary layer.

Many theoretical models involve several different approximations and considerable care needs to be taken to ensure consistency when proceeding beyond the first approximation. The further approximations usually involve wave–current interaction terms.

G. Generation of Currents

The mass transport associated with water waves is of second order in the amplitude but still makes an appreciable contribution to currents in the vicinity of coasts. Details of the mass transport in a uniform wave train are calculated in Longuet-Higgins (1953) and some more recent work is in Sleath (1973, 1974). Mass transport is transformed into a current that is not directly coupled with waves whenever there is wave dissipation. Similarly, when waves gain or lose momentum because of interaction with currents there is a corresponding change in the current. This must be taken into account in any theory dealing with wave–current interactions in water of finite depth unless the waves have infinitesimal amplitude. Equations governing such an interaction are given in Section II,C, but there have been few direct applications to current generation, mostly to longshore currents.

Another mechanism for generating a current from waves is given by Craik (1970). He describes a nonlinear interaction between two wave trains propagating over a depth-dependent shear flow. The idea is that the shear flow models the shear due to wind stress. The current generated has streamlines that superficially resemble those of vortices aligned with the wind. It is suggested that this mechanism may be partly responsible for Langmuir vortices, which are a similar feature observed in the sea.

H. Notation

Notation is generally explained as it is introduced, except for some conventions that are uniform throughout the article. Coordinate axes are chosen with $0z$ vertically upward and $0x$ usually in the direction of the current if it is unidirectional. The plane $z = 0$ is a horizontal surface at or near the mean free surface.

The current field in the absence of waves is $\mathbf{U}(\mathbf{r}, t)$ with components (U, V, W), and it is $\mathbf{U} + \mathbf{u}$ with components $(U + u, V + v, W + w)$ in the presence of waves. Similarly the free surface is

$$z = Z(x, y, t) + \zeta(x, y, t),$$

although Z is zero sufficiently often that the symbol is used with other meanings.

The wave frequency relative to a fixed reference frame is ω and relative to the water is σ. The phase velocity relative to the water is denoted by c in Section II, but elsewhere c is the phase velocity relative to the frame of reference. The wave number vector $\mathbf{k}$ is taken to be $(k, 0, 0)$ or $(l, m, 0)$

depending on circumstances, and θ denotes the angle between $\mathbf{k}$ and $\mathbf{U}$. The amplitude of the wave motion of the water surface is denoted by a.

When tensor notation is used, Greek suffixes have the values 1, 2 and Roman suffixes 1, 2, 3, where

$$(x_1, x_2, x_3) \equiv (x, y, z).$$

The two-dimensional vector operator $(\partial/\partial x, \partial/\partial y, 0)$ is denoted by ∇_1.

In various places one symbol is used with different meanings to avoid using unusual letters or substantial numbers of suffixes. However, when results are cited from other works the notation is changed to agree with that in use in this paper.

II. Large-Scale Currents

A. Introduction

In many instances of waves riding upon currents, the time and length scales determined by the current are many times larger than the period or wavelength of the waves. The natural assumption is to suppose that at any particular point the waves may have the same properties as a plane wave train on a uniform current, and further that the parameters describing the wave train, such as amplitude and wavelength, may vary slowly with the current.

It is intuitively clear that such an approximation is likely to be effective, but the problem can be approached more formally, by requiring

$$k \gg \max\left|\frac{1}{U}\frac{\partial U}{\partial x}\right| \quad \text{and} \quad \omega \gg \max\left|\frac{1}{U}\frac{\partial U}{\partial t}\right|, \tag{2.1}$$

where k and ω are the wave number $(= 2\pi/\text{wavelength})$ and frequency $(= 2\pi/\text{period})$ of the waves. From ratios of such wave and current scales one or more small parameters may be constructed and formal expansions of variables in powers of a small parameter can be used to obtain solutions. It is expected, but not proven, that such solutions are asymptotic to exact solutions.

This short-wave, or large-scale current, approximation has to be used in conjunction with solutions for uniform plane waves on water moving with uniform velocity. Such a flow is irrotational, and hence a velocity potential may be introduced to simplify the analysis, but it is still necessary to approximate to obtain water wave solutions. Some of these approximate solutions are briefly reviewed in Section II,B.

It is commonly the case in short-wave approximations that the solutions are singular on certain lines or points. In water wave examples it is not always clear which approximation is responsible for the singularity. It may be the water wave approximation, e.g., if a small-amplitude assumption is made, or it may be the short-wave approximation of a locally plane wave. For small-amplitude waves the plane-wave approximation can usually be improved in those cases where two possible solutions converge on the same singularity, one representing waves propagating toward it and the other representing waves propagating away from it. The resulting solution will describe waves being reflected. The maximum steepness of such a solution will then indicate whether the water wave approximation is sufficient or not. Specific examples are described in Sections II,D and II,E, but no such solutions have been produced for finite-amplitude waves, although a singularity, which requires such a description, is noted in Section II,E.

The subject of this section is only one aspect of the problem of wave propagation in a slowly varying medium and most work on the subject is not specifically confined to water waves or to moving media. In particular, work published in the last decade on nonlinear short-wave problems, all originating from Whitham's (1965a,b) method of averaging, has shed considerable light on the propagation of linear and nonlinear waves in nonuniform, slowly varying media. The concept of wave action, crystallized by Bretherton and Garrett (1968), is particularly valuable for moving media. An extensive and up-to-date account of the subject is Whitham (1974).

Water waves are also influenced by the depth of water and where this too varies slowly its variation can usually be included. This is done in the rest of this section wherever it is convenient to do so.

B. Waves on Uniform Currents

In many applications of short-wave approximations, either the equations are linear, as in vacuum electromagnetic theory, or a linear approximation is an excellent first approximation, as in acoustics. In water wave problems a linear approximation can be quite sufficient, but it is more usually only a rough guide when waves have an appreciable amplitude and hence greater importance. Hence a few parameters, which are most often used for sinusoidal linear waves, are defined for nonlinear plane waves.

If a wave is periodic in both space and time, then the physical variables describing it will all be functions of a phase

$$\chi = \mathbf{k} \cdot \mathbf{r} - \omega t + \delta, \tag{2.2}$$

in which $\mathbf{k}$, the wave number vector, is perpendicular to planes of constant phase (e.g., wave crests for water waves). Water waves are an example of modal waves, that is, waves that have structure in a dimension in which they do not propagate, in this case down into the water. Thus $\mathbf{k}$ is essentially parallel to the mean water surface and has no component perpendicular to it. The wave number $\mathbf{k}$ and frequency ω are made unique by choosing χ so that its period is 2π; they then correspond to the usual definitions of wave number and radian frequency for sinusoidal waves.

The phase velocity c defined by

$$c = \omega/k, \qquad k = |\mathbf{k}|, \tag{2.3}$$

is also defined for nonlinear waves. It is relevant to note that phase velocity is not a vector, e.g., the phase velocity along a line in the direction of a unit vector $\mathbf{e}$ is $\omega/(\mathbf{k} \cdot \mathbf{e})$.

If $\mathbf{r}$ is a position vector in a frame of reference in which water is moving with uniform velocity $\mathbf{U}$, then the corresponding position vector $\mathbf{r}'$ in a frame of reference moving with the water is given by

$$\mathbf{r}' = \mathbf{r} - \mathbf{U}t. \tag{2.4}$$

Thus a wave on moving water described by

$$f(\mathbf{k} \cdot \mathbf{r} - \omega t) \tag{2.5}$$

is also described by

$$f(\mathbf{k} \cdot \mathbf{r}' + \mathbf{k} \cdot \mathbf{U}t - \omega t) = f(\mathbf{k} \cdot \mathbf{r}' - \sigma t). \tag{2.6}$$

Thus if any wave property (e.g., a dispersion relation) is given for still water for a wave of frequency σ, the corresponding property for a wave on water in uniform motion is given by the relation

$$\sigma = \omega - \mathbf{k} \cdot \mathbf{U}. \tag{2.7}$$

A uniform plane wave train of infinitesimal amplitude, propagating over still water of uniform depth h, with vertical surface displacement

$$\zeta = a \exp[i(\mathbf{k} \cdot \mathbf{r}' - \sigma t + \delta)] \tag{2.8}$$

above the mean level, has velocity potential

$$\phi = \frac{ia\sigma \cosh k(z + h)}{k \sinh kh} \exp[i(\mathbf{k} \cdot \mathbf{r}' - \sigma t + \delta)] \tag{2.9}$$

and dispersion relation

$$\sigma^2 = gk \tanh kh. \tag{2.10}$$

In these expressions the physical quantity is the real part of a complex expression, and z is measured upward from the mean free surface. If surface tension T is to be included, then g should be replaced by $g + Tk^2/\rho$ in the dispersion relation (2.10). The group velocity, the velocity of energy propagation, is

$$\mathbf{c}_g = \frac{1}{2}\frac{\sigma\mathbf{k}}{k^2}\left(1 + \frac{2kh}{\sinh 2kh}\right) \tag{2.11}$$

for gravity waves.

This linear approximation is a good approximation when all three of the parameters

$$ak, \quad a/h, \quad \text{and} \quad a/k^2h^3 \tag{2.12}$$

are much less than one. For finite amplitude waves there are various different approximations, of which two are most relevant. A straightforward perturbation expansion in powers of ak gives the Stokes' wave approximation, which is appropriate for water of moderate or great depth, specifically when a/k^2h^3 is small. For shallow-water waves a more subtle expansion, balancing the effects of a/h against ak, is needed to produce the cnoidal wave solution (e.g., see Whitham, 1974). Such expansion procedures are cumbersome for dealing with the highest waves and usually separate approaches have been made to that problem. However, recently, computer-assisted calculations have enabled expansions to be carried out to high orders giving results for most of the range of possible periodic waves (Schwartz, 1974) and for the limiting case of the solitary wave (Longuet-Higgins and Fenton, 1974). For purely capillary waves Crapper's (1957) exact solution covers the whole range of amplitudes.

Stokes (1847) noted that there is ambiguity in defining "still water" for a finite-amplitude wave train on water of finite depth. The two natural definitions, (i) the average velocity is zero at any point that is always submerged, and (ii) the average flow of water through any vertical plane is zero, are not equivalent. This is clearly shown by considering the most general form of the velocity potential for a periodic wave. Since only the physical variables need be periodic

$$\phi = \boldsymbol{\beta} \cdot \mathbf{r} - \gamma t + \Phi(\chi), \tag{2.13}$$

where χ is the phase (2.2). The constant $\boldsymbol{\beta}$ corresponds to a uniform velocity. The physical interpretation of γ is less clear, but it contributes to the mean pressure and is thus related to the mean level of the water.

If definition (i) for still water is chosen, then $\boldsymbol{\beta} = 0$. On the other hand, with definition (ii),

$$\boldsymbol{\beta} h = -\int_0^{2\pi/\omega} \left[\int_{-h}^{\zeta} \nabla\Phi \, dz\right] dt. \tag{2.14}$$

This value of $\boldsymbol{\beta}$ is often called the mass transport velocity of the waves. The contribution to the integral comes mainly from the region above the lowest value of ζ in the troughs of the waves. Thus for small amplitude waves it is of order a^2 and can often be neglected.

For deep water this ambiguity disappears since the still water at great depth provides a reference frame. However the mass transport is still nonzero.

As an example of a finite-amplitude solution the first terms in a Stokes' wave expansion are

$$\zeta = b + a\left[\cos\chi + \frac{ak(3 - T_0^2)}{4T_0^3}\cos 2\chi + O(a^2k^2)\right], \tag{2.15}$$

$$\phi = \boldsymbol{\beta}\cdot\mathbf{r} - \gamma t + \frac{a\sigma}{k}\left[\frac{\cosh k(z+h)}{\sinh kh}\sin\chi + \frac{3ak\cosh 2k(z+h)}{8\sinh 4kh}\sin 2\chi + O(a^2k^2)\right], \tag{2.16}$$

and the dispersion relation

$$(\sigma - \boldsymbol{\beta}\cdot\mathbf{k})^2 = gk\tanh k(h+b)\left[1 + \frac{(9 - 10T_0^2 + 9T_0^2)}{8T_0^4}a^2k^2 + O(a^4k^4)\right], \tag{2.17}$$

where $T_0 = \tanh kh$. The parameters $\mathbf{k}$, ω, a and $\boldsymbol{\beta}$, γ, b define a specific wave train within a phase shift δ, and the dispersion relation (2.17) provides one equation between them. The choice of a frame of reference determines $\boldsymbol{\beta}$, e.g., definition (ii) for still water gives

$$\boldsymbol{\beta} = \tfrac{1}{2}\mathbf{k}a^2c\coth kh + O(a^3k^2c). \tag{2.18}$$

Either γ or b may be chosen arbitrarily but they must satisfy the relation

$$\gamma = \tfrac{1}{2}\beta^2 + gb + [(1 - T_0^2)\sigma^2a^2/4T_0^2] + O(a^3k\sigma^2) \tag{2.19}$$

obtained from the constant terms in Bernoulli's equation. More details of finite-amplitude waves are given in Section II,C, which shows the advantage of leaving $\boldsymbol{\beta}$, γ, b in these expressions.

Now, consider infinitesimal waves on the surface of a uniform stream $\mathbf{U}$. The dispersion relation (2.10) becomes

$$(\omega - \mathbf{k}\cdot\mathbf{U})^2 = gk\tanh kh \tag{2.20}$$

after using (2.7). This may be conveniently rewritten

$$\omega = \pm\sigma(k) + \mathbf{k}\cdot\mathbf{U}, \tag{2.21}$$

where

$$\sigma(k) = +(gk \tanh kh)^{1/2}. \tag{2.22}$$

A common and direct use of dispersion relations is to find the value of k once ω is known (or vice versa) in order to calculate other wave properties. For example:

(i) Measurements of $\zeta(t)$ at one point may be available and a measure of velocity fluctuations on the bottom may be required; or

(ii) waves may be generated at a fixed frequency ω, as in many experiments.

In the absence of a current, k is determined uniquely but the direction of $\mathbf{k}$ is undetermined. In the $\mathbf{k}$ plane (i.e., a plane where $\mathbf{k}$ is a position vector) the locus of possible solutions is a circle. There is a greater lack of uniqueness in solving Eq. (2.21) for $\mathbf{k}$ when there is a current, even if $\mathbf{U}$ is known.

The easiest way of appreciating the solution of the dispersion relation for $\mathbf{k}$ is to consider the intersection of the plane

$$m = \omega - \mathbf{k} \cdot \mathbf{U} \tag{2.23}$$

with the surface of revolution

$$m = \pm\sigma(k) \tag{2.24}$$

in $(\mathbf{k}, m)$ space. [This is a development of the graphical method of solution given by Jonsson *et al.* (1970) for $\mathbf{k}$ parallel to $\mathbf{U}$.] The general form of the locus of solutions for $\mathbf{U} \neq 0$ is seen by noting that if $\mathbf{k}$ is perpendicular to $\mathbf{U}$ then the current does not affect the solution, while for $\mathbf{k}$ in any other direction, specified by a unit vector $\mathbf{e}$, a diametral section of (2.24) yields a curve as shown in Fig. 1. The trace of a typical plane (2.23) is also shown, and four solution points A, B, C, and D in that diametral plane are labeled.

The solution point A corresponds to waves with a component of $\mathbf{k}$ in the direction of the current, being swept along by it so that the measured frequency ω is greater than the frequency σ relative to the water. Similarly, B represents waves with a component of $\mathbf{k}$ opposed to the current direction traveling more slowly relative to a fixed observer so that ω is less than σ. These solutions effectively exhibit the Doppler effect, with appropriate corrections for dispersion.

The solution represented by point C does not occur without the current, or for nondispersive waves. It corresponds to waves propagating against the current, in the sense that their crests move upstream, but their energy is being swept downstream. That is,

$$-c < U \cos\theta < -c_g, \tag{2.25}$$

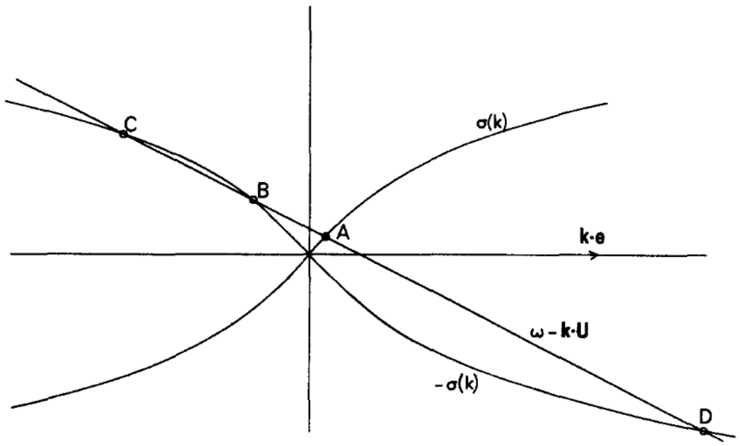


FIG. 1. Solution of the dispersion relation showing multiple values of **k** for given ω, h, and **U**.

where θ is the angle between **k** and **U**. These waves have to be generated on the current.

The point D corresponds to waves with

$$\sigma < 0 \qquad \text{and} \qquad \mathbf{k} \cdot \mathbf{e} < 0. \tag{2.26}$$

They are waves with a phase velocity relative to the water in the $-\mathbf{e}$ direction on a current with a relatively strong component in the $+\mathbf{e}$ direction, that is, waves whose direction of propagation is partly upstream but which are being swept downstream faster than their phase velocity.

For a sufficiently strong current, the solutions B and C may coalesce for some value of θ. It is easily shown that this occurs when

$$c_g + U \cos\theta = 0. \tag{2.27}$$

In these circumstances the wave energy is either at rest or moving perpendicular to the current. This is just the property required of a hydraulic breakwater in order to stop incident waves being transmitted. It is convenient to call such a current velocity a "stopping velocity." Since the phase velocity of gravity waves is always greater than their group velocity the waves' crests would be progressing against the stream but the wave energy would not. This can give rise to a fascinating sight, either in a laboratory flume or on a river, of a small group of waves generated at an appropriate point, to all appearances continually moving but in fact just staying put. If

$$c_g + U \cos\theta < 0, \tag{2.28}$$

then solutions of the types B and C do not exist.

Another interesting case is when $\omega = 0$, that is, waves are stationary on the current although their energy is being swept downstream. The condition for stationary waves may also be written

$$c + U \cos \theta = 0. \tag{2.29}$$

Such waves occur frequently, usually caused by a fixed obstacle or a boundary of the flow.

The difference between the dispersion relation (2.20) for waves on a current and (2.10) for waves on still water can be an appreciable source of error if the presence of a current is overlooked. Such errors are greatest if relations between wave properties at the surface and on the bottom are used, for example, if measurements of $\zeta(t)$ are used to deduce bottom velocities, or if measurements of pressure at the bottom are used to deduce surface wave amplitudes. [Jonsson *et al.* (1970) give one numerical example.]

Taking the latter of these two examples, if a Fourier component of the pressure fluctuation at $z = -h$ has amplitude $p(\omega)$, then the corresponding surface amplitude component is

$$a(\omega) = p(\omega) \cosh kh/\rho g. \tag{2.30}$$

The maximum errors will clearly occur when $\mathbf{k}$ is parallel or antiparallel to $\mathbf{U}$. These are best displayed in dimensionless form.

Introduce

$$\Omega = \omega(h/g)^{1/2} \quad \text{and} \quad F = U(gh)^{-1/2}, \tag{2.31}$$

so that the dispersion relations (2.10) and (2.20) become

$$\Omega^2 = p \tanh p \tag{2.32}$$

$$(\Omega - Fq)^2 = q \tanh q, \tag{2.33}$$

where kh is written p in the case of no current and q when a current is included. Thus if (2.32) is used in error for (2.33), the computed amplitude a_1 and the correct amplitude a_2 are in the ratio

$$a_2/a_1 = \cosh q/\cosh p. \tag{2.34}$$

This has been computed and the relative error $(a_2 - a_1)/a_1$ is shown in Fig. 2. For the case of small F and $\Omega > 1.5$, it is sufficient to use the deep-water approximations for the dispersion relations, and the computation of this error is then a simple exercise.

Inspection of Fig. 2 shows that adverse currents have the greatest effect. As an illustration, Table 1 displays the minimum period of waves for which a current of 0.5 m sec^{-1} can be ignored if errors are to be kept within 5% or within 20%. This velocity is typical of tidal currents in many parts of the

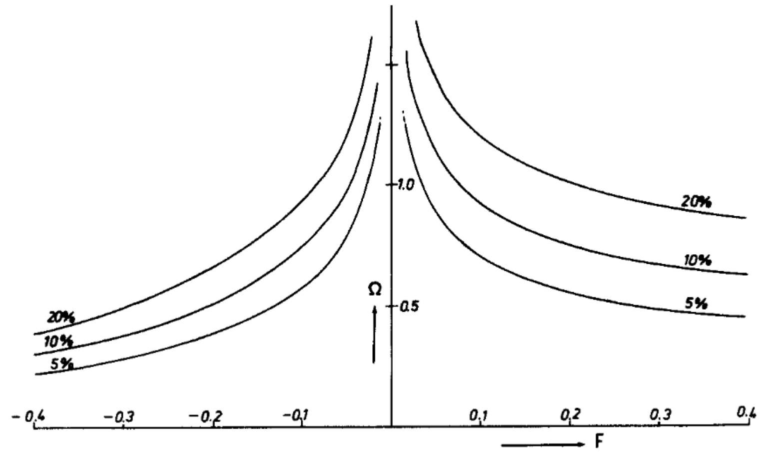


FIG. 2. Relative error in surface wave amplitude calculated from bottom pressures due to ignoring a current component parallel with the wave direction.

world. Most of the combinations of depth and period are within the range that may be measured by this technique. For the shallowest depth given, much of the error comes from the size of the current velocity compared with the wave velocity, whereas at the other extreme the differing variation of pressure with depth is most significant.

TABLE 1

MINIMUM PERIOD OF WAVES FOR WHICH A CURRENT OF 0.5 m sec^{-1} MAY BE IGNORED IN CALCULATING SURFACE AMPLITUDES FROM BOTTOM PRESSURE MEASUREMENTS IF ERRORS ARE TO BE LESS THAN 5 AND 20%

	Depth (m)				
	1	2	5	10	100
Period with error of 5% (sec)	4.5	5.4	6.9	8.0	14
Period with error of 20% (sec)	2.7	3.2	4.3	5.3	11

Simultaneous measurements of bottom pressure and surface elevation are reported by Draper (1957), and a comparison is made with the theoretical ratio, based on still water. Quite a wide scatter is shown. However, most of the results are within the range

$$1.00 < a_2/a_1 < 1.16 \tag{2.35}$$

and can be explained by currents setting against the waves with velocities up to 1.2 m sec^{-1} (i.e., assuming a depth of about 6 m, which is mentioned for the two records shown). The measurements were made on Eastbourne pier. Charts of English Channel tidal streams (Hydrographic Department, 1973) indicate that currents in that region can reach 2.6 knots (1.3 m $\sec^{-1}$) for ordinary spring tides so that the scatter is adequately explained, except for the paucity of records with $a_2/a_1 < 1.00$ corresponding to favorable currents. Maybe measurements were only made at certain states of the tide.

For waves generated by wind, it is a common belief that if the wind is opposed to the current the waves are larger. Naturally this is most simply explained by the greater velocity of the wind relative to the water. However this is insufficient to explain experimental results such as those of Francis and Dudgeon (1967). Vincent (1975) points out that the effective fetch of the wind is increased when there is an opposing current, since the wave energy must travel correspondingly slower. This provides a satisfactory explanation for Francis and Dudgeon's experiments. In particular, it shows that if $U + c_g = 0$ then large wave amplitudes may be found at very short fetches.

There are many other problems of waves on uniform currents, but most of them do not merit attention here since there is no important difference from the corresponding problem for still water. For example, the waves generated by an obstacle in a uniform flow are identical to those generated by the same obstacle moving at constant speed through still water. However, if there are also other waves on the flow that are incident on the obstacle, there may be significant interaction. This is particularly the case where the frequency of the waves is such that eddy shedding by the obstacle is enhanced. Recent experimental work on circular cylinders in a uniform flow by King *et al.* (1973) shows the wide range of frequencies at which vortex induced oscillations do occur, and work reported by Tanida *et al.* (1973) describes some of the effects of oscillating the cylinders. Experimental work to explore the effects of water waves is being initiated in view of its importance to large maritime structures.

C. Waves on Slowly Varying Currents

Large-scale variation of a current can change all the parameters describing a wave train. Some aspects of these changes are relatively simple to interpret. For example, if the flow accelerates or decelerates, the frequency of the waves, ω, will increase or decrease, the well-known Doppler effect. On the other hand, if waves propagate onto a faster or slower flow, the frequency will remain constant, but the wavelength will either increase or decrease. To a certain extent this is due to the extension or contraction of the water surface. If there is a nonzero angle between the wave number vector and the current then a change in current will lead to refraction of the waves.

At first sight it might appear that the consequent amplitude changes might be found by assuming conservation of wave energy. This is not the case; there is a transfer of energy between the current and the waves. However, there is a conserved quantity. It is "wave action" and is equal to the wave energy density divided by the frequency of the waves *relative to the water*.

In commencing a study of waves on slowly varying currents it is natural to start by specifying the current field:

$$\mathbf{U} = \mathbf{U}(\mathbf{r}, t). \tag{2.36}$$

However, in general this is not possible. As is described in Section II,B there is an ambiguity in the definition of still water for finite-amplitude waves, which is usually interpreted as a mass flow associated with the waves. Thus superposition of waves on a current changes the current and its determination becomes part of the problem. Whitham (1962) highlights this aspect of the subject and gives specific examples. There are two important situations where the mass transport has negligible effects on the current field. One is in deep water, and the other is for infinitesimal waves. For these the current field may be specified in advance.

For a consistent approach the current field should satisfy the equations of motion, and this is essential for proving general mathematical results. Much published work assumes either that there is potential flow or that the flow satisfies the finite-amplitude shallow-water wave equations. However, in experimental or natural conditions the current field is usually turbulent and the mean flow $\mathbf{U}(\mathbf{r}, t)$ satisfies no simple set of equations. Whether or not it may be reasonable to neglect the effect of turbulence on the waves is a matter discussed further in Section V. It is likely to be impractical to include it in any detailed analysis, except in those cases where it is sufficiently strongly developed to have a marked dissipative effect.

Although the flows considered in this section are assumed to satisfy the inviscid equations of motion, it is reasonable to apply the techniques described to any flow that is very nearly uniform over a distance of a few wavelengths, a time of a few periods, and to a depth of one wavelength, or to the bottom if that is less deep. To simplify presentation, the free surface of the flow without any waves is assumed to be approximately horizontal. Large-scale flows with appreciable surface slopes and vertical accelerations are considered in Section II,F.

The method that is most effective in describing wave propagation in slowly varying media is the use of an averaged Lagrangian developed by Whitham for nonlinear dispersive waves. [A full account is given in Whitham (1974).] Two other approaches are also valuable; one is direct integration with respect to z and averaging of the equations of motion; the other is to assume an appropriate form for the solution in terms of a perturbation expansion and substitute in the equations of motion. All three have their value but the Lagrangian approach is most informative and is given correspondingly more attention.

1. *The Averaged Lagrangian and Its Application*

A variational formulation of the water wave problem is provided by Luke (1967) for the exact irrotational case, and some indication of how to proceed for rotational flows. Appropriate approximate Lagrangians for long-wave equations are given by Whitham (1967a). An averaged Lagrangian is found by substituting an appropriate plane wave solution, or approximate solution, into the Lagrangian, integrating with respect to depth, and averaging over the phase χ.

The water wave Lagrangians all involve potentials so that appropriate parameters corresponding to β, γ, and b of the Stokes' wave approximations (2.15) and (2.16) need to be included. Thus in general the averaged Lagrangian is a function

$$\mathscr{L}(\mathbf{k}, \omega, a; \boldsymbol{\beta}, \gamma, b; \mathbf{U}, h). \tag{2.37}$$

In particular examples it may be convenient to combine $\boldsymbol{\beta}$ and b with $\mathbf{U}$ and h, respectively.

The primary assumption is that if the wave properties are slowly varying—not only because of variations in $\mathbf{U}$ and h, but also due to slow variations in initial or boundary conditions—then the variational principle

$$\delta\mathscr{L} = 0 \tag{2.38}$$

applies for variations of all the wave parameters. Examples of averaged Lagrangians are given at the end of this subsection.

The derivation of the Euler equations corresponding to the variational principle (2.38) is simplified by reintroducing the phase χ, by the relations

$$\mathbf{k} = \nabla\chi \qquad \text{and} \qquad \omega = -\partial\chi/\partial t, \tag{2.39}$$

and by introducing a pseudophase ψ defined by

$$\boldsymbol{\beta} = \nabla\psi \qquad \text{and} \qquad \gamma = -\partial\psi/\partial t. \tag{2.40}$$

The second derivatives of these phases provide four consistency conditions

$$(\partial\mathbf{k}/\partial t) + \nabla\omega = 0, \tag{2.41}$$

$$\nabla \times \mathbf{k} = 0, \tag{2.42}$$

$$(\partial\boldsymbol{\beta}/\partial t) + \nabla\gamma = 0, \tag{2.43}$$

$$\nabla \times \boldsymbol{\beta} = 0, \tag{2.44}$$

which are sometimes called kinematic relations. There are then four Euler equations corresponding to variation with respect to a, χ, b, and ψ.

The averaged Lagrangians do not include any derivatives of a and b, so that the corresponding Euler equations are simply

$$\partial \mathscr{L}/\partial a = 0, \tag{2.45}$$

$$\partial \mathscr{L}/\partial b = 0, \tag{2.46}$$

and are relations between the parameters of (2.37) involving no derivatives. Indeed, (2.45) corresponds to the dispersion relation and (2.46) to the Bernoulli relation (2.19) between $\boldsymbol{\beta}$, γ, and $\mathbf{b}$.

Since χ and ψ do not appear explicitly in $\mathscr{L}$, the Euler equations for their variations are

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \omega}\right) = \nabla \cdot \frac{\partial \mathscr{L}}{\partial \mathbf{k}}, \tag{2.47}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \gamma}\right) = \nabla \cdot \frac{\partial \mathscr{L}}{\partial \boldsymbol{\beta}}, \tag{2.48}$$

where $\partial/\partial \mathbf{k}$ and $\partial/\partial \boldsymbol{\beta}$ denote the gradient operator in $\mathbf{k}$ space and $\boldsymbol{\beta}$ space, respectively. Equation (2.48) only occurs for finite-amplitude waves in water of finite depth and is found to be an equation for the conservation of mass, including the effects due to changes of level and mass flow associated with the waves.

By defining

$$A = \partial \mathscr{L}/\partial \omega \tag{2.49}$$

to be "wave action density" and

$$\mathbf{B} = -\partial \mathscr{L}/\partial \mathbf{k} \tag{2.50}$$

to be "wave action flux," Eq. (2.47) takes the form

$$(\partial A/\partial t) + \nabla \cdot \mathbf{B} = 0, \tag{2.51}$$

of a conservation equation for wave action. Equation (2.47) is given in Whitham (1965b), but specific attention is drawn to the advantages of using wave action, when considering moving media, by Bretherton and Garrett (1968). A more general definition of wave action, but equivalent to (2.49) where waves are locally plane, given by Hayes (1970), allows the assertion of conservation of wave action in general conservative systems with periodic waves.

For *linear waves* the set of equations to be solved is (2.41), (2.42), the dispersion relation (2.45), and (2.51). The wave action density

$$A = E/\sigma \tag{2.52}$$

and the wave action flux

$$\mathbf{B} = (\mathbf{U} + \mathbf{c}_g)A = (\mathbf{U} + \mathbf{c}_g)E/\sigma, \tag{2.53}$$

where $E = \frac{1}{2}\rho g a^2$ is the wave energy density, σ the frequency of the waves relative to the water, and $\mathbf{c}_g$ their group velocity relative to the water.

The dispersion relation for linear waves has the form

$$\omega = \mathbf{k} \cdot \mathbf{U} + \sigma(k, h), \tag{2.54}$$

so that Eq. (2.41) becomes

$$\left[\frac{\partial}{\partial t} + (\mathbf{U} + \mathbf{c}_g) \cdot \nabla\right] k_\alpha = -\frac{\partial \sigma}{\partial h}\frac{\partial h}{\partial x_\alpha} - k_\beta \frac{\partial U_\beta}{\partial x_\alpha} \tag{2.55}$$

after using (2.42). Another useful equation that may be obtained from the same three equations is

$$\left[\frac{\partial}{\partial t} + (\mathbf{U} + \mathbf{c}_g) \cdot \nabla\right] \omega = \mathbf{k} \cdot \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \sigma}{\partial h}\frac{\partial h}{\partial t}. \tag{2.56}$$

It is most easily obtained by first differentiating (2.54) with respect to time, and its real value lies in the fact that in many problems the right-hand side of the equation is zero.

The solution of problems involving linear waves is eased by the absence of amplitude from Eqs. (2.41), (2.42), (2.54), (2.55), and (2.56). They are not all independent, as is clear from their derivation, but by choosing the most suitable of them it is often a straightforward matter to find ω and $\mathbf{k}$.

Inspection of Eqs. (2.55) and (2.56) shows that their characteristics are given by the lines

$$(d\mathbf{r}/dt) = \mathbf{U} + \mathbf{c}_g. \tag{2.57}$$

If there are no simplifying features as in the examples of Sections II,D and E, numerical methods may be used. In general it is appropriate to calculate these characteristics or "rays," using Eq. (2.55) for the components of $\mathbf{k}$.

Once rays are found, the wave amplitude is then derived from the wave action equation, which can be written

$$\left[\frac{\partial}{\partial t} + (\mathbf{U} + \mathbf{c}_g) \cdot \nabla\right]\left(\frac{E}{\sigma}\right) + \left[\nabla \cdot (\mathbf{U} + \mathbf{c}_g)\right]\left(\frac{E}{\sigma}\right) = 0, \tag{2.58}$$

which expresses the conservation of wave action between rays. The types of solution that may be obtained in various circumstances are illustrated by examples in Sections II,D and E, where analytic solution is possible.

One defect of this type of approximation is that sets of rays may meet and the solution is then singular with the amplitude of the waves becoming infinite. For example, this occurs at a caustic of rays where two sets of rays are tangential to a line in space, and also at a point where $\mathbf{U} + \mathbf{c}_g$ is zero and the rays are tangential to a line in space–time. Such singularities occur at points where the approximation of a locally plane wave is not valid. Typically there is reflection, and a different type of solution, valid in the neighborhood of the singularity, is required. Typical examples are given in following subsections. Even when such solutions are found, it is prudent to check that the parameters (2.12) are within reasonable bounds for infinitesimal theory. Singularities of this general nature also occur for finite-amplitude waves but the details are different, as is shown for one example in Section II,E. So far, no corresponding solution valid near the singularity has been described.

For finite-amplitude waves the dispersion relation includes the amplitude, so that it is not in general possible to solve for $\mathbf{k}$ and ω independently. However, it is possible to use the dispersion relation (2.45), and the Bernoulli relation (2.46), to eliminate one variable each. [For example, Lighthill (1965) eliminates a in discussing two-dimensional deep-water waves and by retaining χ as a variable finds a second-order equation for it.] For the resulting equations there are four sets of characteristics. Two have velocities close to $\pm(gh)^{1/2}$ and correspond to propagation of changes in mean level and flow rate. The other pair of characteristics may be real, in which case their velocities are near $\mathbf{U} \pm \mathbf{c}_g$, or they may be imaginary. (For infinitesimal waves these characteristics are real but coincident.)

Equations with imaginary characteristics are of elliptic type and are effectively unstable for initial value problems. Lighthill (1965) shows that equations for deep-water gravity waves are elliptic and this ties in with other approaches to the stability of a plane wave train (Benjamin, 1967; Phillips, 1967). An unstable wave train does not necessarily break, but a certain class of modulations of the wave train can grow from an infinitesimal initial magnitude until they substantially modify the wave train.

The stability of a wave train is unaltered by a current, and although most applications may be worked out in terms of an initially uniform wave train instabilities should not be ignored. In some examples, their rate of growth may be slow enough for them to be ignored; on the other hand, propagation times may be long enough for instabilities to significantly affect weakly nonlinear waves, for which one might at first sight expect linear theory to be adequate. Numerical integration is also likely to be difficult when the governing equations are unstable.

For capillary–gravity waves on deep water Lighthill (1965) using a third-order approximation shows that characteristics are imaginary except for the parameter range

$$-1 + 2(3)^{-1/2} < Tk^2/\rho g < \tfrac{1}{2}. \tag{2.59}$$

For large-amplitude deep-water gravity waves an approximate Lagrangian (2.69) enables Lighthill (1967) to show that the characteristics become real for

$$kH > 0.679, \tag{2.60}$$

where H is the crest to trough height.

For gravity waves in a finite depth of water, Whitham (1967b) confirmed Benjamin's stability analysis for two-dimensional disturbances by showing that characteristics are imaginary for

$$kh < 1.36. \tag{2.61}$$

The analysis, using a third-order Stokes wave solution, is continued to three-dimensional modulations by Hayes (1973), who indicates that waves are unstable in this approximation for all kh, but that for kh less than 0.5 the region of instability is small and a better approximation should indicate stability.

Hayes (1973) also presents a different, and in some ways simpler, approach to manipulating the equations. By using wave action A as an independent variable, instead of wave amplitude, it becomes simple to eliminate ω. By its definition

$$A = \partial \mathscr{L}/\partial \omega, \tag{2.62}$$

which integrates to

$$\mathscr{L} = A\omega - \mathscr{H}(\mathbf{k}, A), \tag{2.63}$$

where the "constant" of integration is a Hamiltonian. Further, the dispersion relation is now

$$\partial \mathscr{L}/\partial A = 0, \tag{2.64}$$

which is an explicit expression for ω,

$$\omega = \partial \mathscr{H}/\partial A. \tag{2.65}$$

The $\mathbf{k}$ gradient of this expression for ω,

$$\partial^2 \mathscr{H}/\partial A\, \partial \mathbf{k},$$

is a velocity, which Hayes calls the basic group velocity, and it is of value in interpreting the propagation of changes in wave action for a finite-amplitude wave train.

It seems inappropriate to summarize this work in any more detail since applications to waves on currents have yet to appear; the interested reader should refer to the original papers and to Whitham (1974). However, it is appropriate to record the various forms of average Lagrangian that have been evaluated.

For pure capillary waves on deep water, Crapper's (1957) solution gives

$$\mathscr{L} = 2T - \rho\sigma^2/k^3 - T^2k^3/\rho\sigma^2. \tag{2.66}$$

A second-order solution for capillary–gravity waves on deep water gives

$$\begin{aligned}\mathscr{L} = \frac{g}{k^2}\Bigg[&\frac{1-2\kappa}{2\kappa^2+\kappa+8}\left(\frac{\sigma^2}{gk}-1-\kappa\right)^2 \\ &+\frac{24\kappa^5-116\kappa^4-74\kappa^3+351\kappa^2-110\kappa+64}{(2\kappa^2+\kappa+8)^3(3\kappa-1)} \\ &\times\left(\frac{\sigma^2}{gk}-1-\kappa\right)^3+\cdots\Bigg],\end{aligned} \tag{2.67}$$

where

$$\kappa = Tk^2/\rho g, \tag{2.68}$$

and in both (2.66) and (2.67) the amplitude has been eliminated by use of the dispersion relation. These results are from Lighthill [1965; Eqs. (80) and (91)].

For deep-water gravity waves of all amplitudes, Lighthill's (1967) approximation is

$$\mathscr{L} = (\rho g/8k^2)(s^2 - s^3 - s^4), \tag{2.69}$$

with

$$s = \sigma^2/gk - 1, \tag{2.70}$$

and the range of s from small-amplitude to maximum steepness waves is $0 \le s \le 0.2$. To find the trough to crest height, Lighthill suggests the approximation

$$s = k^2a^2, \tag{2.71}$$

where here $2a$ is that height. It would be worth reexamining this useful approximation in the light of the properties of high waves calculated by Schwarz (1974) and Longuet-Higgins (1975) (e.g., the quantity s has a maximum value of 0.193).

For gravity waves on finite depth of water, Whitham (1967b, 1974) finds, from the Stokes' approximation,

$$\mathscr{L} = \rho\left(\frac{1}{2}\beta^2 - \gamma\right)(h+b) + \frac{1}{2}\rho g b^2 + \frac{1}{4}\rho g a^2\left[1 - \frac{(\sigma - \boldsymbol{\beta}\cdot\mathbf{k})^2}{gk\tanh k(h+b)}\right] + \frac{\rho k^2 a^4(9T_0^4 - 10T_0^2 + 9)}{64T_0^4} + \cdots, \tag{2.72}$$

where the symbols are the same as in Eq. (2.15) and thereafter. There is no need for $\boldsymbol{\beta}$ to be small in expression (2.72) and it could be used to represent the total current. In using most Lagrangians for waves on currents, σ would be substituted by $(\omega - \mathbf{U}\cdot\mathbf{k})$.

This method of using an averaged Lagrangian is justified by using a two-time (and length) scale expansion. Further approximations are possible. Chu and Mei (1970) discuss details for Stokes waves, and also (Chu and Mei, 1971) give specific examples, in particular showing how the instability of a uniform wave train develops into stronger modulations that settle down into groups of waves of permanent form. Whitham (1974, Ch. 14 and Sect. 15.5) shows how such an analysis can be derived from the Lagrangian approach. No applications have been given to waves on a variable current, but the solutions discussed in Sections II,D and E indicate that when singularities occur it may be because the basic assumption of a nearly plane wave is at fault, rather than too high a modulation rate.

2. *Averaged Equations of Motion*

A more direct approach to waves on a current is to start with the equations of motion and to divide the velocity and other variables into mean and fluctuating parts. The equations may then be averaged over the phase of the fluctuating motion after integration with respect to z. Then each term needs to be identified with appropriate physical quantities such as the wave energy density E. The method is straightforward in concept but appreciable algebraic manipulation is needed. Details are not presented here, but are given in the book by Phillips (1966). [Mei (1973) gives a minor correction; a term $\bar{p}_d\, \partial d/\partial x_\alpha$ should be added to the left-hand side of his Eq. (3.6.11).]

The advantage of this approach is that it is easier to appreciate the physical significance of terms and hence to make appropriate additions to the equations in order to account for wave dissipation, wave generation, or even wave breaking.

The simplest case is for infinitesimal waves. The consistency conditions

(2.41) and (2.42) and the dispersion relation are again applicable. Instead of the wave action equation an energy equation is derived:

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x}[(U_\alpha + cg_\alpha)E] + S_{\alpha\beta}\frac{\partial U_\beta}{\partial x_\alpha} + 0. \tag{2.73}$$

The term $S_{\alpha\beta}$ is called the radiation stress (Longuet-Higgins and Stewart, 1960) and is given, to a first approximation, by

$$S_{\alpha\beta} = \rho \int_{-h}^{0} \overline{u_\alpha u_\beta}\, dz + \delta_{\alpha\beta} \int_{-h}^{0} \overline{p}\, dz = O(a^3), \tag{2.74}$$

where u_α and p are the velocity and pressure fields associated with the waves. This expression indicates that $S_{\alpha\beta}$ may be considered as a wave momentum–flow tensor, or alternatively as minus a wave Reynolds' stress tensor. Its product with the current gradient in (2.73) gives the rate of energy transfer between waves and current.

The energy equation (2.73) is exactly equivalent to the wave-action conservation equation (2.58). Bretherton and Garrett (1968) show the equivalence for a current satisfying the finite-amplitude shallow-water equations. Most other relevant flows are such that the waves are effectively on deep water, and it is a simple corollary of their proof to show the equivalence for deep-water waves.

For capillary–gravity waves, expression (2.74) for $S_{\alpha\beta}$ is

$$\frac{Ekh}{\sinh 2kh}\delta_{\alpha\beta} + \left[\frac{kh}{\sinh 2kh} + \frac{1+3\kappa}{2(1+\kappa)}\right]E\hat{k}_\alpha \hat{k}_\beta, \tag{2.75}$$

where $\kappa = Tk^2/\rho g$ and $\hat{k}_\alpha$ are the components of a unit vector in the $\mathbf{k}$ direction (Longuet-Higgins and Stewart, 1964). For gravity waves this may be written

$$S_{\alpha\beta} = E\left(\frac{c_g}{c} - \frac{1}{2}\right)\delta_{\alpha\beta} + E\frac{c_g}{c}\hat{k}_\alpha \hat{k}_\beta. \tag{2.76}$$

Radiation stress is a valuable concept when interpreting the generation of longshore currents by waves incident on a coast; see Longuet-Higgins (1972) for a detailed account.

A somewhat different way of identifying terms in the averaged equations of motion is given by Hasselmann (1971). Emphasis is on the Eulerian view for the mass flow associated with the waves, that is, that it occurs only in the surface layer above the wave troughs. Particular attention is given to short waves riding on much longer waves, which are discussed further in Section II,F.

3. *Perturbation Equations and Asymptotic Expansions*

In a sense this title covers nearly all the work discussed in Section II. Most of the plane-wave solutions used result from a perturbation expansion in powers of the wave steepness or similar parameter. The equations for the behavior of the wave trains can also be derived as the first terms of an expansion in powers of k^{-1}, or some other parameter expressing the small size of the waves relative to the scale of the current. Indeed, this section could have been introduced with a formal expansion in terms of ε and k^{-1}. Valuable though such an approach may be in improving the theoretical foundations of the equations, it seems inappropriate in this particular field, where in most instances a first approximation in terms of k^{-1} is all that has been found, and experiment and observation are still needed to assess the value of theoretical results.

The purpose of this subsection is to indicate where a direct approach has been used or could be used with advantage. This is at present confined to infinitesimal waves, whose motion is sufficiently small that it is a perturbation of the current satisfying linear equations of motion. In the body of the water,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{U} + \frac{1}{\rho}\nabla p = 0, \tag{2.77}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.78}$$

with boundary conditions

$$p = \rho g \zeta, \qquad (\partial \zeta/\partial t) + (\mathbf{U} \cdot \nabla)\zeta = w, \tag{2.79}$$

at the mean free surface if the vertical acceleration of the current is negligible (if not, see Section II,F).

An alternative is available if the current velocity field is also very weak. Then one may assume that the two velocity fields are simply additive with any interaction appearing as a second-order perturbation. Radiation stress was first introduced in order to give a physical interpretation of the results of such a perturbation analysis (Longuet-Higgins and Stewart, 1960). The weak-current approach is also used by Taylor (1962) and can be of considerable value since results can often be found without any assumptions about the scale of the current. In principle it is possible to solve any problem, but in practice the solutions are often singular, demonstrating that the implicit assumption of a weak interaction is not always tenable.

Ideally one would like to solve Eqs. (2.77) and (2.78), with their boundary conditions (2.79), exactly for particular flows. Then no approximation about the scale of the current field is required. The only such solutions the author is aware of are for some flows of the form $\mathbf{U} = (U(z), 0, 0)$ (Section IV,B) and for stationary waves on a uniform flow bounded by vortex sheets (Section III).

Once it is assumed that the current varies on a much larger scale than the waves, examination of the perturbation equations and boundary conditions (2.77)–(2.79) shows that the only term involving derivatives of **U** is the third one in (2.77). Thus, any first approximation does not include it, and **U** then appears in the equations in the same way as for a uniform velocity. This suggests the most usual way of finding solutions, which exactly parallels the results described in Section II,C,1. A solution for a uniform current is chosen, such as a plane wave, and the parameters describing that solution, such as a and k, are thought of as functions, varying slowly with **U**. Typically they are expanded in an asymptotic series of the form

$$a = a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \cdots, \tag{2.80}$$

and after substitution in the perturbation problems, coefficients of each power of k are set equal to zero. The actual details can vary appreciably and multiple-length scales may be introduced.

If a plane wave is chosen as the assumed form of solution (or "ansatz"), then the first approximation gives an "eikonal" equation for the rays and the second approximation a "transport" equation for the amplitude, corresponding to the wave action equation. An advantage of this approach is that for a velocity field that can be expressed in simple analytic terms higher approximations in k^{-1} can be readily found.

The major advantage of choosing an "ansatz" and substituting it in the perturbation equations is in dealing with cases where the wave field is not locally a plane wave. (For example, however many terms in k^{-1} are taken, it is not possible to describe any reflection if the initial assumption is that there is a single wave train.) In particular, this approach can be used to describe regions where rays form a caustic, and the plane-wave approximation gives a singular solution for the amplitude.

At least two types of caustic may be identified. If the rays forming the caustic are straight in its neighborhood, then the caustic is similar to a caustic in a uniform medium and is not directly affected by the current field. At the other extreme, if the caustic is caused entirely by the curvature of rays, it is the refracting effect of the current that is the prime cause of its existence. In this case the wave properties at the caustic depend on the value of the current and on its gradient. (Note that where the ray theory gives a good approximation the wave properties depend only on the value of the current.) Two very different examples of this latter class of caustic are discussed in Sections II,D and E.

It is instructive to note the analogy with short-wavelength solutions of the linear second-order ordinary differential equation,

$$(d^2u/dx^2) + N(x)u = 0. \tag{2.81}$$

When $N(x) \gg 0$, sinusoidal solutions with slowly varying amplitude are found by the well-known WKB (or Liouville–Green) method. These correspond to the ray solutions in our problems. The exponential solutions for $N(x) < 0$ can be found in a similar manner and correspond to regions where waves do not penetrate. These are joined near points $N(x) = 0$ by connecting solutions in terms of Airy functions (e.g., see Lanczos, 1961). Such points correspond to caustics and solutions in the neighborhood of a simple caustic usually involve Airy functions, though not necessarily in the relatively direct form that is found for equations of the form (2.81).

Similarly, when the function $N(x)$ has two zeros within a reasonably short distance, parabolic cylinder functions are appropriate for approximating solutions. Two examples of the corresponding problem, when the ray solutions indicate two adjacent caustics, are mentioned in Section II,E.

Another wave pattern that needs special treatment is a focus of rays. There is no simple analogy with ordinary differential equations. Most foci form as cusps of caustics, and Pearcey (1946) gives an appropriate "cusp" function together with numerical values. Uniform asymptotic expansions for a cusped caustic are discussed by Ludwig (1966). If rays form a perfect focus (i.e., all crossing at the same point) a solution of the form

$$\sum_n a_n r^n J_n(\cos n\theta) e^{i\omega t} \tag{2.82}$$

may be used, and the asymptotic form of the Bessel functions used to match with the ray approximation at a sufficiently large value of r. It is particularly easy to find the amplitude at the geometric focus, $r = 0$, with expression (2.82), but Pearcey's cusp function shows that the maximum wave amplitude is not usually at that point. A large-scale current distribution can cause waves to form different types of caustic. It is possible that there might be different types of focus.

Waves patterns where the ray approximation gives a singularity in the amplitude, such as caustics and foci, have often been interpreted as regions where water waves break. This interpretation is only justified if a local approximation gives a large amplitude, and more significantly a high wave steepness, in its vicinity.

Following this general discussion, the next two subsections consider two very simple examples of unidirectional flow. A variety of solutions can be found analytically, which illustrate some of the effects that can happen in a more complex problem. There appear to be very few examples published where more realistic flows have been tackled. The change in period of ocean waves, measured in Cornwall, due to tidal currents encountered in their passage over the continental shelf is one example calculated by Barber (1949) and is in satisfactory agreement with measurements.

D. Steady Current, Varying with Distance along the Stream

A current of the form

$$\mathbf{U} = [U(x), 0, 0] \tag{2.83}$$

can describe the flow in channels and rivers where the velocity varies in response to the depth $h(x)$ and slope of the bottom. The flow from a hydraulic breakwater may also be described in this way, although in practice such a flow is best confined to a surface layer, and (2.83) is then only a reasonable approximation for very short waves.

A closely related flow can occur with a propagating wave, when

$$\mathbf{U} = [U(x - Ct), 0, 0]. \tag{2.84}$$

If C is constant this flow is of the type (2.83) in a frame of reference moving with velocity C. Examples are the surface currents due to internal waves or to long shallow-water waves. The form (2.84) may describe tidal currents, but more usually the tide is made up of more than one propagating mode. However, in all these possible applications it is likely that the basic long-wave motion is of sufficiently small amplitude that different modes may be superposed.

The simplest example is when waves travel perpendicular to the current, that is,

$$\mathbf{k} \cdot \mathbf{U} = 0. \tag{2.85}$$

There is no interaction.

A simple, less trivial example is when waves travel parallel to the current, so that

$$\mathbf{k} \cdot \mathbf{U} = \pm kU. \tag{2.86}$$

For definiteness we take $\mathbf{k}$ in the positive x direction so that if waves propagate against a current U is negative. Initially, attention is restricted to infinitesimal waves.

Since the current is steady, or considered in a reference frame in which that is so, Eq. (2.56) implies that on rays

$$dx/dt = U + c_g, \tag{2.87}$$

$$\omega = \text{const.} \tag{2.88}$$

The waves must vary slowly for the theory to apply so that there is little loss of generality in considering a steady state situation in which ω is constant everywhere; the dispersion relation

$$\sigma^2 = (\omega - Uk)^2 = gk \tanh kh \tag{2.89}$$

is then the only equation needed to determine k and hence σ.

The ambiguities in solution of Eq. (2.89) mentioned in Section II,B are avoided by requiring continuity of solution as U and h vary. For numerical solutions this suggests that the numerical integration of a differentiated form of (2.89) is likely to be preferred to direct solution, though care should be taken near double roots. Explicit solutions can be found for deep-water waves, in which case it is usually more convenient to work with the phase velocity of the waves relative to the water, that is,

$$c = \sigma/k = +(g/k)^{1/2} = g/\sigma, \tag{2.90}$$

which with (2.89) gives

$$\omega = k(c + U) = g(c + U)/c^2 \tag{2.91}$$

as the equation for c.

The conservation of wave action implies that the wave action flux

$$B = \rho g a^2(U + c_g)/2\sigma \tag{2.92}$$

is constant and equal to its value at the point where the waves are generated. For deep-water waves this becomes

$$a^2 = 2B/\rho c(U + \tfrac{1}{2}c), \tag{2.93}$$

and there is no difficulty in finding the amplitude.

Even where the depth is significant in the dispersion relation there is no difficulty in calculating the wave properties numerically. The results of such calculations are presented by Jonsson *et al.* (1970) in both tabular and graphical form for the case of steady channel flow, where

$$U(x)h(x) = Q. \tag{2.94}$$

Expression (2.92) indicates that the amplitude may become unrealistically large if either σ becomes very large or $U + c_g$ approaches zero. The former case does not arise for waves generated on still water, but can do so for waves initially on a current. Since large σ implies very short waves the deep-water relations hold, and (2.90) shows that σ large corresponds to $c \to 0$. Typical curves representing Eq. (2.91) in the (U, c) plane clearly show that $c \to 0$ only as $U \to 0$ (Fig. 3). That is the situation in which

$$-U > c_g, \tag{2.95}$$

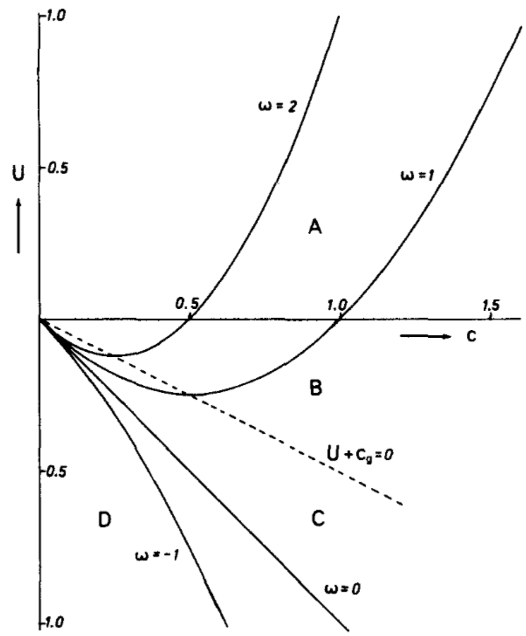


FIG. 3. Relation (2.91) between c and U for deep-water waves for different values of ω. Units are chosen with $g = 1$. The letters A, B, C, and D in the sectors shown refer to the four solutions in Fig. 1.

and the waves are being swept downstream although they propagate upstream relative to the water. As the water slows down the waves continually shorten and increase in steepness so that small-amplitude theory becomes invalid. In practice the waves break and little or no energy is propagated into still water.

The other singularity when

$$U + c_g = 0 \tag{2.96}$$

is more interesting. At such a point the wave number and frequency are both finite, and if a subscript 0 indicates still-water values and subscript 1 conditions at (2.96), then

$$U_1 = -\tfrac{1}{2}c_1 = -\tfrac{1}{4}c_0\,, \qquad k_1 = 4k_0\,, \qquad \sigma_1 = 2\sigma_0\,. \tag{2.97}$$

The velocity U_1 is conveniently called the "stopping" velocity since for waves propagating against a stream it has the effect of stopping them. It is behind the idea of a hydraulic breakwater and can also be observed in rivers and in "tide races," which occur off headlands or in channels where tidal currents are locally enhanced. Such a stopping velocity leads to very rough water surfaces as the wave energy density increases substantially. Upstream of such points, especially if the current slackens, the surface of the water is especially smooth as all short waves are eliminated. As (2.97) shows, the current required is only one-quarter of the phase velocity for the waves in still water, because the waves have a smaller wavelength.

Most accounts of this phenomena (e.g., Phillips, 1966) assume that, because of the singularity in amplitude, waves necessarily break before reaching the stopping velocity, finite amplitude effects only serving to give a different value to U_1. This is not necessarily the case since the stopping velocity corresponds to a caustic of rays. There is a corresponding set of rays being swept downstream. It is thus possible for the waves to be reflected and swept downstream, as shorter waves, by the current. Details of this process are found by looking for a solution of the perturbation equations (2.77)–(2.79) in the neighborhood of the stopping velocity.

Specifically, by using a perturbation stream function of the form

$$\psi(x, z)e^{-i\omega t}, \tag{2.98}$$

the equations can be reduced to

$$\mathscr{K}\nabla_1^2\psi = 0 \qquad \text{in} \quad z < 0, \tag{2.99}$$

with the boundary condition

$$\mathscr{K}^2\, \partial\psi/\partial z = g\, \partial^2\psi/\partial z^2 \qquad \text{on} \quad z = 0, \tag{2.100}$$

where the operator

$$\mathscr{K} = U(x)\frac{\partial}{\partial x} + \frac{dU}{dx} - i\omega. \tag{2.101}$$

To solve these equations near the stopping point it is necessary to choose a simple form for $U(x)$ that has a value and a gradient at that point that equal those for the actual velocity. Two possible choices are

$$U(x) = -\tfrac{1}{2}c_1(1 + \beta x) \tag{2.102}$$

and

$$U(x) = -\tfrac{1}{2}c_1 e^{\beta x}. \tag{2.103}$$

For both of these functions it is relatively straightforward to find the complex Fourier transform with respect to x of Eq. (2.99) and its boundary condition (2.100). If U were constant ($= U_1$), then the Fourier transform of ψ would be

$$\Psi(k, z) = \delta(k_1)e^{kz}. \tag{2.104}$$

For small βx it is expected that the solution will be a perturbation of this solution, so it is assumed that the only relevant values of the transform variable k are those such that

$$\kappa = (k - k_1)/k_1 \ll 1. \tag{2.105}$$

This assumption simplifies Eq. (2.99) to Laplace's equation and gives the form

$$\Psi(k, z) = A(k)e^{kz}. \tag{2.106}$$

With the further, short-wave assumption that

$$\beta \ll k_1, \tag{2.107}$$

the boundary conditions resulting from either (2.102) or (2.103) can be simplified to a first-order differential equation in k for $kA(k)$. This can be integrated to give, near $\beta x = 0$,

$$\psi = \text{const} \times \int_{-\infty}^{\infty} \exp\{k_1(ix + z) + i\beta^2 k_1 \kappa^3/12 + \kappa[k_1 \beta(ix + z) - \beta/4]\}\, d\kappa, \tag{2.108}$$

which may be rewritten in terms of the Airy function,

$$\psi(x, y) = A \exp[k_1(ix + z)]\ \mathrm{Ai}[(4k_1^2\beta)^{1/3}(x - iz + i/4k_1)]. \tag{2.109}$$

This is a local solution for the wave motion, valid near the stopping point. It has no singularity, indicating that reflection can occur. However, to be useful it needs to be matched to an outer solution corresponding to the *two* branches of Eq. (2.91) for c and the solution of (2.93) for the amplitude with B positive for waves traveling upstream, but with B of the same magnitude and negative for the reflected waves. Such matching can be done by using the asymptotic expansion of the Airy function. Since solution (2.109) is valid when

$$\beta x \ll 1, \tag{2.110}$$

and the asymptotic expansion applies when

$$(4k_1^2\beta)^{1/3}x \gg 1, \tag{2.111}$$

there is a matching region of overlap where

$$\beta^{-3} \gg x^3 \gg (4k_1^2\beta)^{-1}, \tag{2.112}$$

which is nonzero because of (2.107).

The result, in terms of the amplitude a_0 on still water, is

$$A = 4\pi^{1/2}(2k_1/\beta)^{1/6}a_0\,, \tag{2.113}$$

and the amplification in wave steepness is

$$0.536Ak_1 = 17.1(k_1/\beta)^{1/6}a_0 k_0\,, \qquad (2.114)$$

where 0.536 is the maximum value of the Airy function. This indicates that although the small-amplitude approximation may be uniformly valid, it is so, for waves starting on still water, only for extremely gentle initial waves, since even if the large parameter k_1/β is not very big the factor 17.1 is appreciable.

Further light is shed on the matter by Fig. 4, which shows the variation of k, a, and ak with a logarithmic scale, according to the simple ray theory. The ak curve for the reflected waves is particularly interesting. It has a minimum value where $U = 0.98U_1$ corresponding to an amplification of 33.6. This implies that any reflected waves continually increase in steepness as they are swept further away from the stopping point. Indeed, it is likely that the minimum will be obscured by the reflection process.

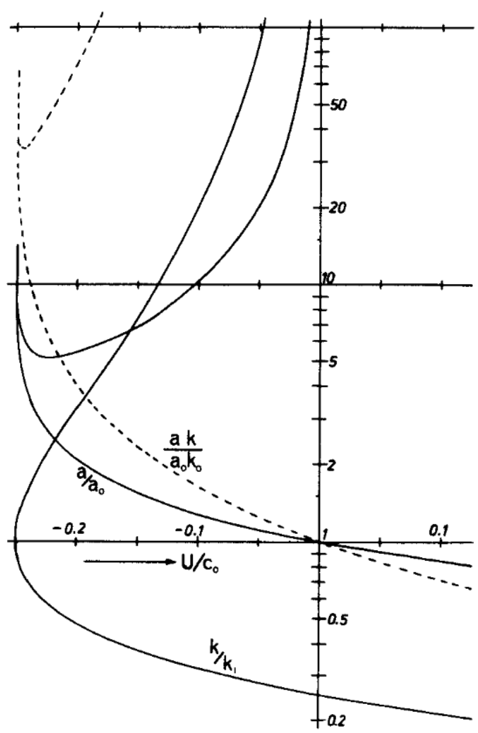


Fig. 4. Variation of wave number, amplitude, and wave steepness for deep-water waves on a current, according to the ray theory, Eqs. (2.91) and (2.93). A subscript 0 refers to still-water conditions and subscript 1 to conditions at the stopping velocity, $U_1 = -\frac{1}{4}c_0$.

Reflection has not been reported by experimenters with hydraulic breakwaters. The reason is easy to see. The reflected waves cannot propagate to still water and have a shorter wavelength. For example, at a point where $U = 0.75U_1$, the ratio of wavelengths, incident to reflected, is 9 to 1. Such short waves are unlikely to be recognized as a direct reflection without prior knowledge.

Because of the substantial amplification, all wave trains eventually become too steep to be described by an infinitesimal wave theory if they meet substantial adverse currents. In practice they will break either before, at, or after reflection. Finite-amplitude effects can alter the stopping velocity. Experimental results for hydraulic breakwaters (e.g., Evans, 1955, Fig. 5) indicate an appreciable variation of stopping velocity with wave steepness, so that it is valuable to have some theoretical results for finite-amplitude waves.

The result

$$\omega = k(c + U) \tag{2.115}$$

still holds for finite-amplitude waves. In deep water we know that the dispersion relation has the form

$$c^2 = (g/k)[1 + f(a^2k^2)]. \tag{2.116}$$

Thus

$$c^2 = v_0[1 + f(a^2k^2)](c + U). \tag{2.117}$$

where $v_0 = g/\omega$ is the velocity of infinitesimal waves of frequency ω on still water. For a *given* value of $f(a^2k^2)$, Eq. (2.117) has coincident roots for c if

$$U = -\tfrac{1}{4}v_0[1 + f(a^2k^2)]. \tag{2.118}$$

Thus the maximum magnitude of U is given by substituting the maximum value of $f(a^2k^2)$ in (2.118). This is 0.193 (Schwartz, 1974), and the corresponding minimum stopping velocity is

$$U_1 = -0.298v_0\,. \tag{2.119}$$

By using Lighthill's approximate Lagrangian (2.69) the analysis may be carried further. A number of numerical solutions obtained from it are given by Crapper (1972). The function $f(a^2k^2)$ in (2.116) is replaced by s so that

$$c^2 = v_0(1 + s)(c + U). \tag{2.120}$$

The wave action flux $\partial\mathscr{L}/\partial k$ can be found explicitly in terms of s and c, and the requirement that it be constant gives

$$2v_0(c + U)^3P(s) + c(c + U)^2(c + 2U)P'(s) = 2v_0c_0^3P(s_0) + c_0^4P'(s_0). \tag{2.121}$$

in which

$$P(s) = s^2 - s^3 - s^4 \tag{2.122}$$

and a subscript 0 refers to conditions where $U = 0$. Although Eqs. (2.120) and (2.121) may be solved for the unknowns s and c, (2.121) is sufficiently complex to obscure interpretation.

However, by using the argument leading to Eq. (2.118) we can investigate whether waves break before meeting a stopping velocity. That is, choose s at the stopping velocity, s_1 say, then

$$U_1 = -\tfrac{1}{4}v_0(1 + s_1), \tag{2.123}$$

$$c_1 = -2U_1. \tag{2.124}$$

Substitution of these values in the left-hand side of Eq. (2.124) then gives an equation for the corresponding value of s_0, which is easily solved since for values of s_1 less than or equal to that for the highest wave, s_0 is small. Indeed, for $s_1 = 0.193$, $s_0 = 0.00067$. For small s,

$$s = a^2k^2, \tag{2.125}$$

so that $a_0 k_0 = 0.0258$ in this case. The implication is that for greater values of $a_0 k_0$, waves must break before a stopping velocity is reached, and that for smaller values, waves may be reflected before they break.

Another simple result is obtained if s_1 is also very small, in which case (2.121) simplifies to

$$s_1^2 = 64s_0 \qquad \text{or} \qquad a_1 k_1 = 8(a_0 k_0)^{1/2}. \tag{2.126}$$

Experimentally, it is found that the greater the still-water steepness of waves, the greater is the velocity required to stop them. This is in apparent contradiction to the above analysis. However, experimental waves have been sufficiently steep that they can be expected to break before the stopping velocity, and details of the breaking process are likely to affect the result. For example, the momentum lost from the waves in breaking is transferred to the current, and although this does not make a significant contribution to the mean current in deep water it will certainly affect the current distribution in the surface layer that directly influences the waves. Effects of a variation of mean velocity with depth are examined in Section IV.

Holliday (1973) has calculated a few solutions for finite-amplitude capillary–gravity waves. These show a considerable variation from gravity waves near stopping points. This is not unexpected because of the variation of c_g relative to c; however, the paper contains no discussion of this.

If waves are at an angle to the current, that angle is an extra variable in the problem. Let θ be the angle between $\mathbf{k}$ and $\mathbf{U}$. The assumed current distribution (2.83) introduces no asymmetry, so that only the range $0 \leq \theta \leq \pi$ need be considered. As already mentioned there is no interaction at $\theta = \pi/2$ so that the analysis proceeds most naturally by considering $0 \leq \theta < \pi/2$. There is no loss of generality as long as both positive and negative values of $U(x)$ are considered.

The extra equation needed to determine θ comes from the second consistency condition (2.42), which integrates immediately, since all the wave parameters are independent of y, to give

$$k \sin \theta = m. \tag{2.127}$$

For steady wave conditions, or following waves along a ray, m is a constant. Again the frequency

$$\omega = k(c + U \cos \theta) \tag{2.128}$$

is constant, and the dispersion relation may be used. The wave action flux **B** is also constant: the x component, which is the most relevant, is

$$B_x = \rho g a^2 (U + c_g \cos \theta)/2\sigma \tag{2.129}$$

for infinitesimal waves.

By limiting consideration to deep-water waves it is easy to eliminate k and c from Eq. (2.127), (2.128), and the dispersion relation to find an equation for θ:

$$U^*(x) \cos \theta = \sin \theta - (M \sin \theta)^{1/2}. \tag{2.130}$$

In this equation U^* is a dimensionless velocity,

$$U^*(x) = U(x)m/\omega. \tag{2.131}$$

The velocity ω/m is the phase velocity of the waves in the y direction. The constant

$$M = gm/\omega^2 \tag{2.132}$$

and equals the value of $\sin \theta$ for the waves where $U = 0$. However, M may be greater than one if the waves are generated on moving water and cannot propagate onto water at rest.

Once θ has been found from Eq. (2.130) it is straightforward to find the corresponding values of k, c, and a. In order to illustrate the behavior of (2.130), a graphical method of solution is illustrated in Fig. 5. Each side of Eq. (2.130) is plotted against θ for a number of values of M and U^*. A given set of waves may be followed by looking along a line corresponding to the right-hand side of the equation.

For waves generated on still water it is apparent that as they propagate onto a current with a component in the same direction, θ increases and their

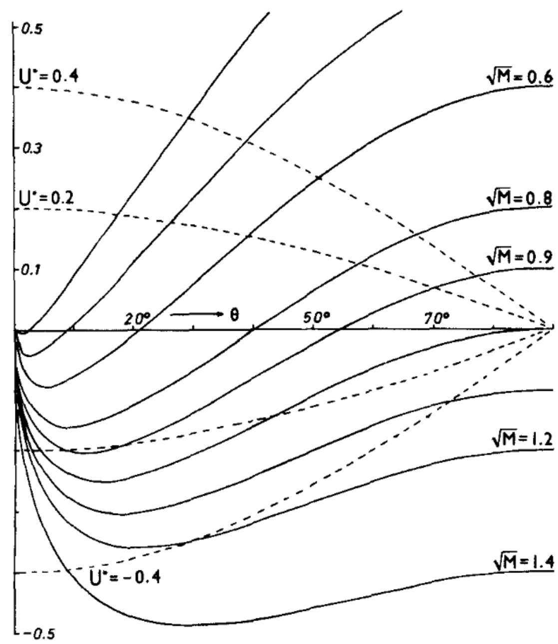


FIG. 5. The two families of lines correspond to the left- and right-hand sides of Eq. (2.130). Their intersections give values of θ that solve the equation for the appropriate values of U^* and M.

wavelength, from Eq. (2.127), also increases. The opposite happens when they propagate onto an adverse current, and there is a stopping velocity such that

$$U + c_g \cos\theta = 0. \tag{2.133}$$

In Fig. 5 this corresponds to a double point of intersection of the two appropriate curves.

The behavior near the stopping point is similar to that already described for $\theta = 0$. That is, if the waves are sufficiently gentle they are reflected into waves that are swept downstream. In this case the value of θ diminishes toward zero for the reflected wave train. Alternatively the waves become sufficiently steep for nonlinear effects to become important and may break. Crapper (1972) gives the results of three numerical calculations for this problem, using Lighthill's approximation. No unexpected behavior is apparent.

There are three solutions for θ of Eq. (2.130) in the parameter range

$$3\sqrt{2}/4 > M > 1, \qquad -\sqrt{2}/4 < U^* < 0. \tag{2.134}$$

Hence there are two ways in which a double root can arise. The new type, which only occurs in this range, corresponds to waves that are being swept downstream to a stopping point. This can happen to waves that initially are propagating almost directly across the stream. If they are swept downstream into slower moving water they are refracted until their component of $\mathbf{c}_g$ upstream is large enough to allow

$$U + c_g \cos\theta = 0. \tag{2.135}$$

At this stopping point they may be reflected upstream and will continue to be refracted toward the direction from which the stream comes. If the adverse current increases sufficiently upstream, a normal stopping point may be reached. A second reflection may occur and the waves are once more swept downstream. Now the crests would be nearly perpendicular to the stream and the wave number substantially increased. Figure 6 gives a sketch

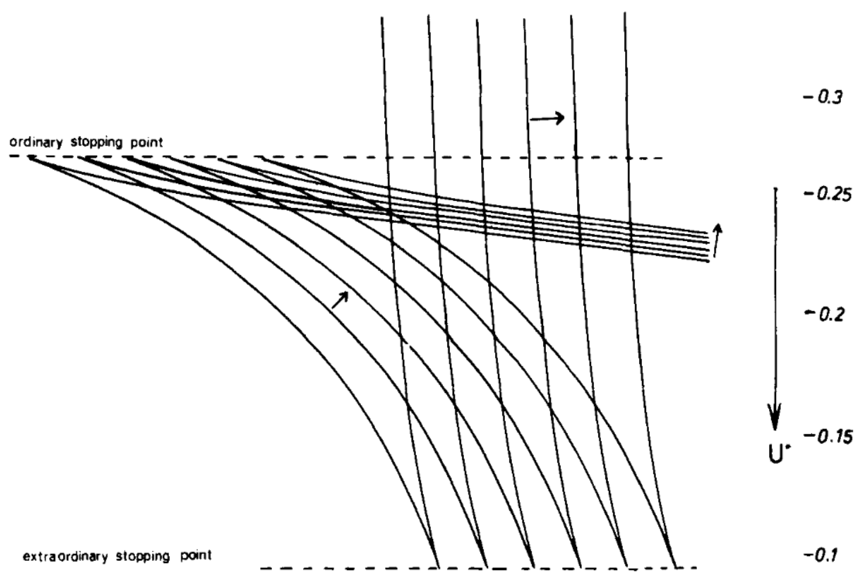


FIG. 6. A sketch solution for the wave crests of a wave train, with $M = 1.02$, reflected at two stopping points. The arrows denote the direction of propagation of the waves (i.e., $\mathbf{c}_g$, not the ray direction $\mathbf{c}_g + \mathbf{U}$).

of what wave crests may look like for such waves on a stream with a constant velocity gradient dU/dx. The rays do not conveniently fit on the same diagram. They are tangent to the lines of crest cusps. As is usually the case for nonzero currents, the rays are not perpendicular to the wave crests; indeed, for

$$\tan\theta = -mU/\omega = -U^*, \tag{2.136}$$

the rays are parallel to the wave crests. In Fig. 6 this occurs at a velocity just below the normal stopping velocity for the rays reflected at that point.

The preceding analysis is clearly relevant to waves on rivers and flows in channels. Waves on such flows are usually generated either by wind, by ships or boats, or by obstacles. The latter waves are stationary and this a special case, with $\omega = 0$. The other two generating mechanisms generally form waves directly on the current and thus not necessarily with the parameters relevant to still-water conditions.

The effects near the stopping velocity are readily observed, and result in steep waves as the theory indicates. An account of waves generated by a launch becoming steeper as they propagate upstream and hence giving an unusual sequence of events to an observer is given by Peregrine (1972). For example, a launch traveling upstream at constant velocity relative to the water, against a current with velocity increasing with distance upstream, generates waves. To an observer stationed at a point below the stopping velocity, the waves appear to increase with time after the boat's passage. This is easily explained by the theory at the beginning of this subsection.

For sea waves, the group velocity of significant waves is usually much greater than current velocities except for regions of especially strong currents near coasts. The effects of such currents producing tidal races at headlands have already been mentioned. Other strong currents are found at the entrance to harbors, rivers, and enclosed bays or lagoons. Again the effects of currents on incident waves, lengthening or diminishing their wavelength, is often clear. Johnson (1947) gives two aerial photographs showing the effect of an opposing and a helping current.

Laboratory experiments to investigate the phenomenon, reported by Hales and Herbich (1972), showed strong effects. However, their photographs indicate that the current was too narrow for a theory based on large-scale current variations to be applicable. This also appears to be the case for rip currents, which occur near beaches.

Rip currents are the strongest of the currents generated in, and confined to, the region just outside and within the surf zone on beaches. They are jetlike currents, usually returning toward the sea the mass flow associated with the incident waves. Quite often their position is stabilized by minor beach features, such as longshore bars, but they can and do occur on uniform beaches. There are a number of papers trying to account for the generation of rip currents, especially those on uniform beaches. For example, Noda (1974) shows how the interaction of waves and minor beach topography gives rise to strong rip currents; in a following paper (Noda *et al.* 1974) a partially successful attempt to include the effects of the currents on the waves demonstrates how important this is. Arthur (1950) gives a computed ray diagram showing the effect of a current over a uniformly sloping bottom. Le Blond and Tang (1974) use a more complete approach, including the whole cycle: incident waves generate a current that in turn influences the waves. Their method is to look for a perturbation to a steady two-dimensional solution.

However, the width of rip currents is rarely much greater than the wavelength of the dominant incident waves. Thus, as Arthur (1950) acknowledges, the assumption of a slowly varying wave train is inappropriate. Indeed, whereas one might expect a more rapid increase in incident wave amplitude on the rip current, observations indicate appreciably less growth in wave amplitude. For example, a descriptive paper by Shepard *et al.* (1941) comments that a gap in the breakers often occurs at the site of a rip current. As this subsection shows, it is possible for the waves to be reflected by the current, but this seems a little unlikely. The results of Section II,E indicate that the lateral variation of current ought to concentrate the wave energy near the current's center. It seems more likely that the waves are diffracted away by the current and a completely different approach is required (e.g., see Section III). In practice the currents are usually unsteady, and the bed topography also needs to be considered.

Although the more dramatic effects of the interaction between waves and currents require relatively strong currents, the more usual currents of seas and oceans do exert an appreciable influence on waves. For such weaker currents, analysis is simplified by assuming all changes to be small and then using the differentials of Eqs. (2.89) and (2.92). For deep-water waves this results in

$$d\sigma/2\sigma = dk/k = -2\,dc/c = dU/(U + \tfrac{1}{2}c), \qquad (2.137)$$

$$da/a = -(3U + 2c)\,dU/(2U + c)^2. \qquad (2.138)$$

Explicit results like this may also be found when the depth of water influences the wave's velocity.

This type of approach is particularly appropriate to currents associated with long traveling waves of the form (2.84). The most commonly encountered long shallow-water waves are the tides. The above formulas can easily be used where waves and tidal current are approximately unidirectional, even if the tide is not simply a progressive wave. A standing wave component in the tide can be written as the sum of two progressive waves, e.g.,

$$2A \cos Kx \cos \Omega t = A \cos K(x - Ct) + A \cos K(x + Ct), \qquad (2.139)$$

where $C = \Omega/K$, and the contribution due to each of these is added together. The velocity U in (2.137) and (2.138) is equal to minus the phase velocity of the long wave, and dU equals the variation in velocity. This case of waves on tidal currents is examined in detail by Vincent (1975). It is readily shown that wave amplitudes are amplified most at high tide, for unidirectional flows. Vincent shows that statistics derived from wave measurements in the southern North Sea are consistent with this result.

For long internal waves results (2.137) and (2.138) still hold, but there is an interesting complication. Although the currents involved are weak, the phase velocity C of an internal wave is such that it is possible for

$$C = c_g . \tag{2.140}$$

That is, effects associated with stopping velocities may occur, for surface waves traveling in the same direction as an internal wave. Gargett and Hughes (1972) report regular surface markings in a region of strong internal wave activity, which on closer inspection are regions of steep, long-crested, short waves. The paper contains two photographs of them as well as a theoretical analysis. The various stopping velocities for waves at an angle to the current are identified, and their importance is discussed. However, their physical nature and detailed solutions are not found. The waves look as if they are associated with a stopping velocity.

Experiments on long internal waves with short gravity waves propagating in the same direction are reported by Lewis *et al.* (1974). They particularly investigated conditions near the stopping velocity. In the frame of reference moving with the internal waves, the conditions of the laboratory experiment, where both wave trains are generated at the same point, do not correspond to a steady state, so the analysis of this subsection does not directly apply. The paper presents an analysis of a perturbation about the basic state. This indicates that the situation in which the group velocity of the surface waves equals the phase velocity of the internal waves may be considered a resonant interaction. Detailed measurements are presented that are in good agreement with the theoretical results.

For many purposes sea waves are best considered in terms of their energy spectra, and the transformation of spectra by currents has been discussed by Phillips (1966, p. 60), Huang *et al.* (1972), Vincent (1975), and Gargett and Hughes (1972), although the latter is only a brief discussion in connection with internal waves.

Huang *et al.* (1972) give the most extensive discussion. They take an empirical form of energy spectrum $E(\omega)$ for wind waves, assume that the waves are generated in still water, and then use relationships equivalent to (2.91) and (2.92) to calculate the corresponding energy spectrum after propagation onto a current. For adverse currents this results in a cutoff at high frequencies corresponding to those waves which cannot propagate upstream. For waves that are being actively generated by the wind, these higher frequencies will be in the "saturated" part of the spectrum (Phillips, 1966, Sect. 4.5). On the other hand if the waves propagate onto a current traveling in the same direction this part of the spectrum will be appreciably diminished. Huang *et al.* suggest that since the high wave number part of the spectrum contributes most to surface roughness, this may be used as the basis for a method of measuring surface currents. They fail to note, as Phillips (1966) points out, that an adverse current will keep the spectrum saturated, so that significant changes will be essentially due to the change in current from its maximum adverse value. For ocean currents there is the added complication that waves are generated by the wind, not only in regions of no current but also where there are currents. This is aggravated by the variable nature of most currents since the time history of both waves and currents becomes important, for example, in Barber's (1949) calculation of the changes in wave period, due to tides, of swell from distant storms crossing the area of the continental shelf southwest of Britain.

Some idea of the practical importance of such changes may be obtained from Tung and Huang (1973), although a number of simplifying assumptions are made in their analysis, which is a sequel to Huang *et al.* (1972). A "force" spectrum is deduced for the forces exerted on an obstacle fixed to the sea bed, e.g., a coastal structure or oil rig. For example, using a wave spectrum corresponding to a generating wind speed of 18 m $\sec^{-1}$ (40 mph), their calculations show (in their Fig. 4) that an adverse current of 1 m $\sec^{-1}$ (3.3 ft $\sec^{-1}$) increases the force spectrum maximum by over three times, whereas a simple superposition of waves and current doubles this maximum.

The angular spread of a wave spectrum is also an important parameter. Inspection of Fig. 5 shows that an adverse current $\mathbf{U}$ tends to concentrate the wave spectrum around the $-\mathbf{U}$ direction, while a favorable current tends to widen the angular spread for waves generated on still water.

E. Steady Current, Varying across the Stream

A current of the form

$$\mathbf{U} = (U(y), 0, 0) \tag{2.141}$$

is the simplest form of shear flow. For any function $U(y)$ it is a solution of the inviscid equations of motion for water bounded by a horizontal free surface and a bottom with depth variation $h(y)$. The real flows that support gravity waves are turbulent, but even so form (2.141) can be a reasonable approximation in appreciable portions of the flow field. The results derived from this simple form can be used to interpret real flows. This discussion is confined to steady wave trains on flows of deep water.

If θ is the angle between the wave number vector and the current, we again have that

$$\omega = \sigma + k \cos \theta\, U \tag{2.142}$$

is constant. The consistency condition (2.42) gives

$$k \cos \theta = l, \tag{2.143}$$

another constant, and the y component of wave action flux gives a third constant,

$$B_y = \rho g a^2 c_g \sin \theta / 2\sigma, \tag{2.144}$$

which with the dispersion equation is sufficient to determine the waves.

Equations (2.142) and (2.143) immediately give

$$\sigma = \omega - lU. \tag{2.145}$$

k and c follow from the dispersion relation, and substitution for k in (2.143) gives

$$\cos \theta = gl/(\omega - lU)^2. \tag{2.146}$$

Clearly, for a range of values of U this expression can have a magnitude greater than one. For those velocities there are no waves with parameters ω and l. The critical velocities bounding the region without waves are

$$U = \frac{\omega}{l} \pm \left(\frac{g}{|l|}\right)^{1/2}. \tag{2.147}$$

At this velocity $\theta = 0$ or π, so the waves travel parallel to the current. From the equations for the rays

$$dx/dt = U(y) + c_g \cos \theta, \qquad dy/dt = c_g \sin \theta, \tag{2.148}$$

it is easy to show that they generally have nonzero curvature at the critical velocity. Thus the rays are tangent to a caustic line at such points.

The conservation of wave action gives

$$a^2 = 8lB_y/\rho g \sin 2\theta. \tag{2.149}$$

The amplitude becomes unreasonably large as θ approaches 0, $\pi/2$, and π. We have already noted that the values 0 and π correspond to a caustic, and hence it is a place where the approximation of a plane-wave train becomes invalid. The case $\theta \to \pi/2$ is where waves are refracted so much that the wavelength becomes very short and the small amplitude approximation is no longer valid since the waves are too steep. This latter effect shows more strongly in the expression for the wave steepness

$$a^2k^2 = 4l^3B_y/\rho g \sin \theta \cos^3 \theta. \tag{2.150}$$

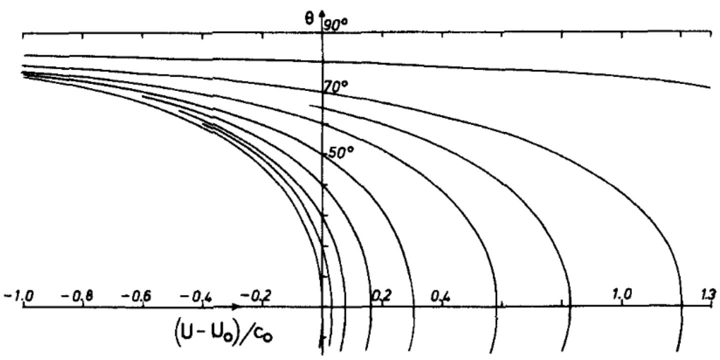


FIG. 7. Curves showing the amount of refraction of wave trains on a shearing current. The angle θ is plotted against the dimensionless velocity change U^*.

The behavior of wave trains can be followed graphically with the aid of Figs. 7 and 8. The amount of refraction is found from Eq. (2.146), which can be cast into the form

$$\cos\theta = \cos\theta_0/(1 - U^* \cos\theta_0)^2, \tag{2.151}$$

where the wave train initially has a phase velocity c_0 and is traveling at an angle θ_0 to a stream U_0. The dimensionless change in velocity

$$U^* = (U - U_0)/c_0 \,. \tag{2.152}$$

Figure 7 shows curves giving θ as a function of U^* for a comprehensive

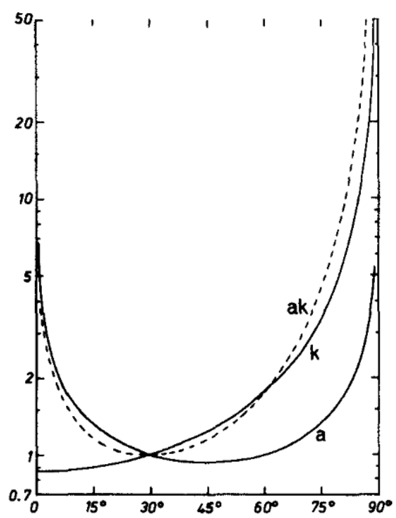


FIG. 8. The variation in amplitude, wave number, and wave steepness for a wave train refracted by a parallel shear flow.

range of values of θ_0. The corresponding variations of amplitude, wave number, and steepness, *ak*, are shown in Fig. 8 on a logarithmic scale. The curves are normalized by the value of each quantity at $\theta = \pi/6$, which is the angle at which *ak* is least. The singularities at both ends of the range of θ are apparent. However, reference to Fig. 7 shows that, starting with some "middling" value of θ, the change in current required to reach $\theta = 0$ is much less than that required to approach $\pi/2$. Thus caustics may occur more often than extremely steep waves due to refraction.

There are uniformly valid small-amplitude approximations including caustics. Such solutions are presented in both McKee (1974) and Peregrine and Smith (1975). McKee considers the case of a shear flow with a monotonic velocity distribution $U(y)$ in the neighborhood of the caustic. Peregrine and Smith consider stationary waves (i.e., $\omega = 0$) on a velocity distribution that has a maximum value. This leads to waves being trapped between two parallel caustics. In all these parallel-flow problems there is no loss of generality in considering stationary waves since the phase velocity in the x direction ($\omega/k \cos\theta$) is constant; thus there is always a frame of reference in uniform translation in which a wave train is stationary.

These two papers differ a little in the form of solution assumed, but for the first approximation in powers of k^{-1} there is no significant difference, except that McKee (1974) allows for a depth variation $h(y)$. The solution has the expected form in that it depends directly on an Airy function. If the caustic is taken to be at $y = y_c$ and $U(y)$ increases with y so that the region with waves is $y < y_c$, then

$$\zeta = A(k\sigma/c_g l)^{1/2}(dY/dy)^{-1/2}\,\mathrm{Ai}(-Y), \tag{2.153}$$

where for $y < y_c$,

$$Y^{3/2} = \frac{3}{2}\int_y^{y_c} k \sin\theta\, dy, \tag{2.154}$$

and k, σ, θ, and c_g take the values that may be deduced from Eqs. (2.142) and (2.143) and the dispersion relation.

The asymptotic expansion of the Airy function shows that the waves are perfectly reflected and that the solution is consistent with Eq. (2.144) for the flow of wave action if

$$A^2 = 8\pi l B_y/\rho g. \tag{2.155}$$

Close to the caustic, this means that (2.153) may be written

$$\zeta = (8\pi\sigma B_y/\rho g c_g)^{1/2}(c_g l/2U')^{1/6}\,\mathrm{Ai}[(2l^2U'/c_g)^{1/3}(y - y_c)], \tag{2.156}$$

where all the parameters and $U' = dU/dY$ are evaluated at the caustic.

McKee's (1974) result incorporating the effect of varying depth is obtained by replacing U' with

$$U' + \frac{1}{l}\frac{\partial \sigma}{\partial h}\frac{dh}{dy}. \tag{2.157}$$

Some simplification is obtained for deep water, since at the caustic

$$c_g = 2\sigma/l. \tag{2.158}$$

In this case the maximum amplitude is

$$0.536A(2\sigma/U')^{1/6}, \tag{2.159}$$

and the corresponding maximum steepness is l times the same expression. Since the waves have a minimum amplitude at $\theta = \pi/4$, the maximum amplification is calculated to be

$$1.065(\sigma/U')^{1/6}, \tag{2.160}$$

and the maximum amplification of steepness from its minimum at $\theta = \pi/6$ is

$$0.496(\sigma/U')^{1/6}. \tag{2.161}$$

Clearly, large amplification near a caustic is not likely. For example, the largest values of the parameter σ/U' are likely to occur for sea waves, but even then it is unlikely to exceed 10^6, which gives a maximum steepness less than 5 times the minimum. Thus in many instances small-amplitude theory will be valid at such caustics.

The natural way to consider a caustic after reading the above is to think in terms of a stronger current refracting waves until they are propagating in the current direction and reflected. An alternative view of the same wave system is to consider waves propagating with a component of their direction being upstream. Then a less adverse current may refract them to form a caustic. This is particularly likely to happen on a river or similar stream of water where the flow has a central maximum and slackens toward the edges. When such flows are sufficiently rapid they can sustain stationary waves, and in such circumstances the two caustics, one on each side of the maximum velocity, are easily seen by a casual observer. This configuration is considered in detail by Peregrine and Smith (1975). A sketch of the rays is given in Fig. 9a.

When the caustics are far apart, the analysis is similar to McKee's (1974) except for the matching of the solutions for the two caustics. This gives an eigenvalue problem for the wave number l, which is found in the first approximation from

$$\left(N + \frac{1}{2}\right)\pi = \int_{y_1}^{y_2} k \sin\theta \, dy, \tag{2.162}$$

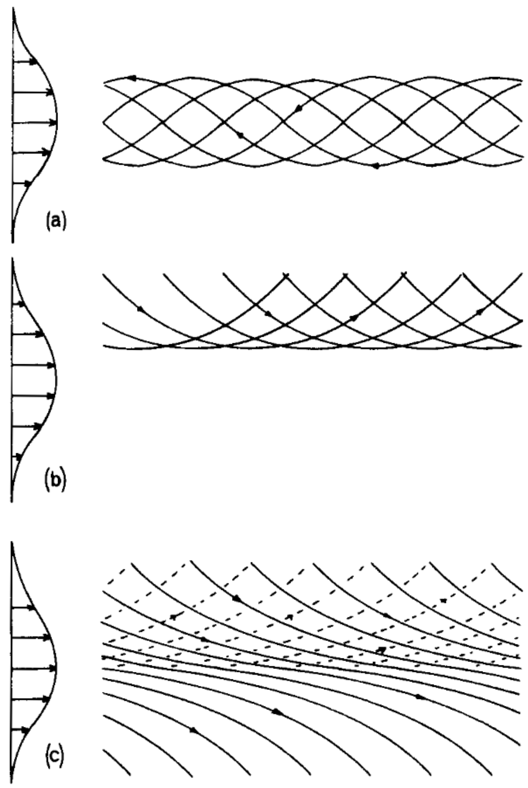


FIG. 9. Ray diagrams for waves encountering a jetlike flow. A velocity profile is at the left-hand side of each diagram. (a) Waves trapped in the region of maximum velocity. (b) Waves excluded from the region of maximum velocity. (c) Waves partially reflected by a "near caustic."

in which y_1 and y_2 are the positions of the caustics and N the number of zeros of the amplitude between them. Peregrine and Smith (1975) give a further approximation and solve the case where N is small, the two caustics are close together, and the appropriate ansatz involves Hermite functions.

Examination of Fig. 7 shows that if the range of θ across the stream is relatively small (e.g., $\pm 15°$) then the waves exist only on velocities very close to the maximum. Thus a line of waves trapped between caustics on a stream of water gives a very meager indication of the actual velocity distribution across the stream.

The converse situation, where waves are refracted away from a stream traveling in their own direction, is examined by McKee (1975). In this case, the waves are outside the region between the two caustics; a ray diagram is sketched in Fig. 9b. If these two caustics are close together, some of the wave action incident on one is reflected while the rest is transmitted. The ansatz is in terms of parabolic cylinder functions (Hermite functions are a relatively simple set of these).

McKee's (1975) results indicate that the reflection coefficient decreases from unity for widely spaced caustics to $2^{-1/2}$ for the case when ray theory indicates that they coincide and the maximum stream velocity equals that at a caustic. It is unlikely that the reflection drops abruptly to zero for a slightly lower maximum current; thus if a ray diagram indicates a "near caustic" some reflection is likely. A near caustic is sketched in Fig. 9c. It is a line of inflection points in rays where the angle between the rays and the inflection point line is small. This may be investigated with the same analysis as McKee (1975).

The diagrams in Fig. 9 also partly illustrate the behavior of waves on a current such as a river. The shear flow refracts waves that are propagating downstream out of the region of maximum velocity. On a river this increases the dissipation and scattering of such waves by the river's banks. Conversely, waves propagating upstream are refracted toward the center of the stream and as a result suffer little scattering or dissipation. This is especially evident for wind-generated waves, which even propagate upstream around corners into reaches sheltered from the wind. Similarly in these circumstances winds opposing currents will be able to generate larger waves than comparable winds in the current direction. However, in most rivers this latter effect will be obscured by the greater effective fetch available for the upstream wind.

Finite-amplitude effects may be worked out for deep-water waves, away from any caustics, by Lighthill's approximate Lagrangian (2.69). Crapper (1972) shows the results for a few representative initial conditions but because of his computation method fails to note that there is a singularity in the solution differing from that which occurs at a caustic for infinitesimal waves.

Equations (2.143) and (2.145) still hold, so that using

$$1 + s = \sigma^2/gk \tag{2.163}$$

instead of the linear dispersion relation gives

$$\cos\theta = gl(1 + s)/(\omega - lU)^2 = \alpha(y)(1 + s). \tag{2.164}$$

Note that $\alpha(y)$ is the value (2.146) of $\cos\theta$ for small-amplitude waves; thus one effect of appreciable wave steepness is that θ is diminished on a given current for the same parameters l and ω.

The relevant equation for constant wave action flux is

$$\partial\mathscr{L}/\partial m = \text{const} \tag{2.165}$$

where

$$m = k \sin \theta. \tag{2.166}$$

After substitution of $\mathscr{L}$ this may be reduced to

$$\sin \theta \cos^3 \theta \, s(2 + s - 9s^2 + 6s^3) = 2\beta, \tag{2.167}$$

where β is a constant. Equations (2.164) and (2.167) are two equations for β and s. For a given wave train, β is constant but α is a function of y; thus if curves given by (2.164) are drawn in the (θ, s) plane for a range of values of α, then the appropriate curve (2.167) immediately shows the variation of both s and θ.

Curves for representative α and β are shown in Fig. 10. The velocity U

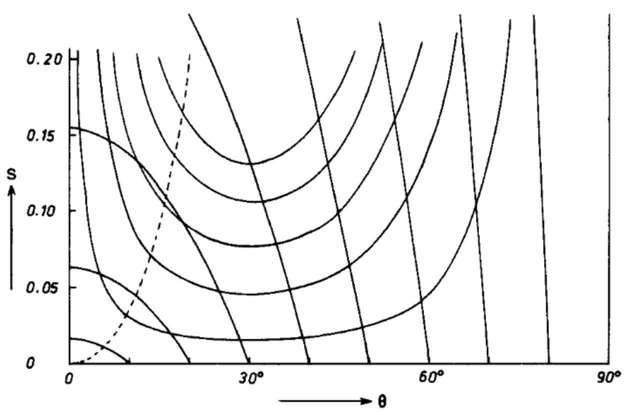


FIG. 10. Solutions for the wave steepness ($s \simeq a^2k^2$) and direction of propagation of finite-amplitude deep-water waves on a shear flow. The lines $\alpha(y) = $ const may be identified by the value of θ at $s = 0$, and use of Fig. 7 then gives the corresponding value of $U(y)$. The lines $\beta = $ const do not touch either axis, and the dashed line is where lines of the two families touch.

corresponding to different values of α can be found from Fig. 7 since $\alpha = \cos \theta$ for zero amplitude. An immediate and striking result is that there are no solutions for velocities corresponding to sufficiently small values of θ. Each curve $\beta = $ const touches one of the family of α curves and does not meet any of the α curves between that point and the origin. At each of these points, joined by a line in Fig. 10, $ds/d\alpha$ (and hence ds/dU and ds/dy) approaches infinity, even though s is finite and may be quite small. The solution for s as a function of U has two branches, which in the (s, U) plane are represented by a single smooth curve. This means that for given ω, l, and U there may be two solutions (s, θ).

The conclusion to be drawn from the singularity of the s derivatives is that the plane-wave assumption does not hold. There is a caustic, but the singularity differs from that for infinitesimal waves. Observation of shear flows does not reveal any tendency for waves to become unduly steep near caustics, so it is likely that the reflected waves generally correspond to solutions on the lower branch of the s, U curve.

The difference from the infinitesimal case is quite striking. For example, for $s = 0.11$ and a velocity that causes infinitesimal waves to be at 30° to the current, the finite-amplitude solution stops with $\theta = 16°$. The difference is still there as $s \to 0$. The line of double points is

$$s = a^2k^2 = \theta^2, \tag{2.168}$$

near the origin.

Thus, given a sufficiently small value of dU/dy, even the gentlest waves may meet this singularity an appreciable distance from the zero-amplitude caustic position. However, one is reassured by the existence of uniformly valid approximations for infinitesimal waves reflecting at a caustic.

This singularity is not an artifact of Lighthill's approximate Lagrangian since it occurs for arbitrarily small s for which any finite-amplitude approximation should give the same result. It is also very similar to a singularity in s arising from an initial-value problem investigated by Lighthill (1965, Sects. 13–15). It seems likely that a similar singularity may be found in caustics caused by varying depth. In that case there is an exact linearized solution for edge waves, which provides a starting point for a uniform finite-amplitude approximation, and, hence, a possible way of gaining further understanding of the problem.

There has been little experimental work on this problem. One of the major problems in experiments is to set up a uniform current with shear since most large-scale flows are turbulent and not unidirectional. Hughes and Stewart (1961) measured the propagation of capillary–gravity waves across a shear flow. Their elegant approach to the turbulence problem was to set up a stable Couette flow. The measurements of wave slope confirm that energy is not conserved and the results are consistent with inclusion of the effects of radiation stress. [Longuet-Higgins and Stewart (1964) point out that the effects of radiation stress are underestimated since an incorrect expression was used.]

An interesting application of refraction analysis is given by Kenyon (1971). The propagation of waves across the Pacific Ocean from storms in the Antarctic Ocean was measured by both Munk *et al.* (1963) and Snodgrass *et al.* (1966). In the former paper, it is noted that "wave inferred directions [to storm centers] are to the left of the location from weather maps," and in the latter paper, measurements were obtained "though the stations are totally shadowed." Kenyon suggests, and gives some detailed figures, that the refraction of waves by the Antarctic circumpolar current is sufficient to account for these anomalies.

Abnormally large waves off the southeast coast of South Africa have been reported over a number of years, but with increasing frequency over the last decade. For example after meeting a large wave the 28,000-ton tanker World Glory broke in two in 1968, and the 260,000-ton ore/oil tanker Svealand was severely damaged by a wave in 1973. Some of these incidents are reported in *Marine Observer* published by the Meteorological Office, and Mallory (1974) gives details of several incidents together with meteorological and hydrographic information.

Mallory's report shows that these exceptional wave conditions occur off an almost straight coastline stretching in a southwesterly direction, with a relatively narrow continental shelf of between 10 and 30 km width. The Agulhas current runs in a southwesterly direction, bounded by the outer edge of the shelf (200-m contour), with its maximum current of over 2 m sec^{-1} just seaward of the shelf edge. Its width is 95–160 km, but it does not usually flow on the shelf, where a countercurrent of up to 1 m sec^{-1} flows when a cold front passes.

The abnormal waves are all observed on the Agulhas current, and their occurrence is either coincident with or a few hours after the passage of a cold front through the area. The southwesterly winds behind the cold front have a fetch of over 1200 km and have been blowing for more than 24 hours. The waves generated in this long fetch will have developed on the westward-flowing Antarctic circumpolar current, but meet the Agulhas current head on, resulting in an increase in amplitude of around 25% on the linear theory for uniform wave trains.

A further increase in amplitude is to be expected because of the jetlike nature of the current. Waves are concentrated by refraction onto the region of maximum velocity and most of the wave energy is confined to this region by bounding caustics. This fits in well with the observations reported by Mallory of higher waves outside the continental shelf on the current. Thus the current and wind systems combine to give a very high level of wave activity. The abnormal waves encountered have usually been much greater in amplitude and length than the general level. However, in any sea the largest waves occur with small probability.

One aspect of caustics that needs investigating in this context is the behavior of a short group of waves traversing a caustic region. This may be a suitable way of modeling the real sea waves, which are rarely coherent for more than a few wavelengths. It seems likely that a solution would show just one or two large waves persisting for a limited time.

Ocean current and wave systems of this sort merit further study since it should prove possible to forecast them and advise shipping accordingly.

Also such waves may occur elsewhere in similar conditions; Casey (1974) in a brief letter mentions an incident off Ushant, where tides would cause the currents, and off Japan.

F. Flows with Significant Vertical Accelerations

The foregoing theory does not apply directly to flows in which the water has appreciable vertical acceleration or surface slopes. Examples of this type of flow are surface gravity waves on deep water and the flow over a waterfall or weir. The former example is of most interest, and the behavior of short waves riding on long waves has been discussed by several authors; it was in Longuet-Higgins and Stewart's (1960) paper on this subject that the concept of radiation stress was introduced.

In that paper a careful perturbation analysis for short waves riding on long waves is presented. Both sets of waves are assumed to have small amplitude and the interaction occurs in second-order terms. Once the appropriate amplitude variation is described, a physical interpretation is discussed, including among other things the idea of radiation stress. It is seen that the effect of the vertical acceleration of the water on the dispersion relation needs to be taken into account. At first sight this appears inconsistent, since all other derivatives of the long-wave velocity field are ignored. In that example, a confirmatory check is available from the perturbation analysis. A more general illustration of the effect of vertical acceleration is given here.

For simplicity of presentation consider the basic flow to be steady and two dimensional; this includes periodic plane waves that are steady in a frame of reference moving with their phase velocity. Introduce orthogonal curvilinear coordinates (s, n) near the water surface such that s is measured along the surface and n increases along normals outward from the surface. Let the basic flow have (s, n) velocity components (U, V) and pressure P satisfying the equations of motion. Then the boundary is $n = 0$, and the boundary conditions are

$$V = 0 \qquad \text{and} \qquad P = \text{const.} \tag{2.169}$$

This flow together with infinitesimal waves riding on it may be described by velocity components $(U + u,\ V + v)$, a pressure $P + p$, and the free surface $n = \eta$. These satisfy linearized inviscid equations of motion:

$$\frac{\partial u}{\partial t} + \frac{1}{(1+\kappa n)}\frac{\partial}{\partial s}(Uu) + \frac{\partial}{\partial n}(Vu) + \kappa uV + \kappa Uv + \frac{1}{\rho(1+\kappa n)}\frac{\partial p}{\partial s} = 0, \tag{2.170}$$

$$\frac{\partial v}{\partial t} + \frac{1}{(1+\kappa n)}\frac{\partial}{\partial s}(Uv) + \frac{\partial}{\partial n}(Vv) - 2\kappa Uu + \frac{1}{\rho}\frac{\partial p}{\partial n} = 0, \tag{2.171}$$

together with linearized boundary conditions applied at $n = 0$:

$$\frac{\partial \eta}{\partial t} + \frac{U}{(1 + \kappa n)} \frac{\partial \eta}{\partial s} = v, \tag{2.172}$$

$$p = -\frac{\partial P}{\partial n} \eta. \tag{2.173}$$

In these equations κ is the curvature of the free surface of the basic flow, with $\kappa > 0$ if the center of curvature is within the fluid.

The equations of motion for the basic flow give

$$-\partial P/\partial n = (DV/Dt) - \mathbf{g} \cdot \hat{\mathbf{n}}, \tag{2.174}$$

where $\mathbf{g}$ is the gravitational field and $\hat{\mathbf{n}}$ the unit vector normal to the free surface. Thus, taking (2.169) into account the boundary condition (2.173) may be rewritten

$$p = (-\mathbf{g} \cdot \hat{\mathbf{n}} - \kappa U^2)\eta. \tag{2.175}$$

Flow along the surface streamline will satisfy Bernoulli's equation so that U is of the order $(gL)^{1/2}$, where L represents the length scale of the basic flow. This is the case for deep-water long waves since U is then approximately equal to the phase velocity, which is $(gL/2\pi)^{1/2}$, where L is the wavelength. If the radius of curvature κ^{-1} is also of order L, then the terms κU^2 and $\mathbf{g} \cdot \hat{\mathbf{n}}$ are of the same order of magnitude and should both be included in the first approximation.

For a small-amplitude deep-water long wave κ^{-1} is of order A/L^2, where A is the wave amplitude, and κU^2 is of order $(A/L)\mathbf{g} \cdot \hat{\mathbf{n}}$. For small A/L it might be assumed negligible. However, this is not the case since the variations in U that are of interest are also of order $(A/L)U$. Thus the term κU^2 should again be included.

Typical terms in the equations of motion are

$$U\, \partial u/\partial s \simeq O(Uuk),$$

$$u\, \partial U/\partial s \simeq O(UuL^{-1}),$$

$$\kappa U u \simeq O(UuL^{-1}).$$

For these terms, the primary assumption that the waves are short compared with L clearly indicates that velocity gradients and the curvature of streamlines need not be included in a first approximation. Thus the dispersion relation for short waves is

$$\sigma^2 = (\omega - kU)^2 \simeq (-\mathbf{g} \cdot \hat{\mathbf{n}} + DV/Dt)k = g^*k, \tag{2.176}$$

where g^* is the "effective gravity" in a frame of reference moving with the free surface of the large-scale flow.

For a steady situation it is reasonable to suppose that wave action is conserved and that the wave action flux

$$B = E(U + c_g)/\sigma \tag{2.177}$$

is constant, where

$$E = \tfrac{1}{2}\rho g^* a^2. \tag{2.178}$$

However, as Bretherton and Garrett (1968) point out, there is ambiguity possible in defining E, the "perturbation energy density" in moving reference frames. This is made apparent by Longuet-Higgins and Stewart (1960) where the form

$$E = \tfrac{1}{4}\rho g^* a^2 + \tfrac{1}{4}\rho g a^2 \tag{2.179}$$

is chosen for waves on a basic flow with small surface slopes, the second term representing potential energy in the gravitational field only. The choice (2.178) seems more appropriate since it corresponds to equipartition of kinetic and potential energy densities. This choice is also supported by consideration of the Lagrangian for a perturbed flow derived by Bretherton and Garrett, Eq. (4.19), although it should be noted that this involves energy per unit horizontal area rather than per unit area of mean free surface.

The solution of these equations differs from that for the simpler flows of Section II,D only in the substitution of g^* for g, so that most results carry over directly once $g^*(s)$ and $U(s)$ are prescribed. However, the example of short waves riding on long waves traveling in the same direction involves a different solution from that discussed in Section II,D.

In a frame of reference in which the long waves appear stationary, the current is

$$U(s) = -C \cos \psi + U^*(s), \tag{2.180}$$

where C is the phase velocity of the long waves, $\psi(s)$ their surface slope, and $U^*(s)$ the water velocity due to the waves. That is, the short waves in this frame of reference are meeting an adverse stream of magnitude C. The phase velocity of the short waves is much less than C, so that

$$\omega = k(c + U) \tag{2.181}$$

is negative, showing that c is always less than $-U(s)$. Similarly the wave action flux B is also negative, and the solution illustrated in Fig. 4 does not represent this situation.

For the case $g^* = g$, the variation of a, k, and ak relative to their values at $U = -2c$ are shown in Fig. 11. A large range of velocities are shown in the

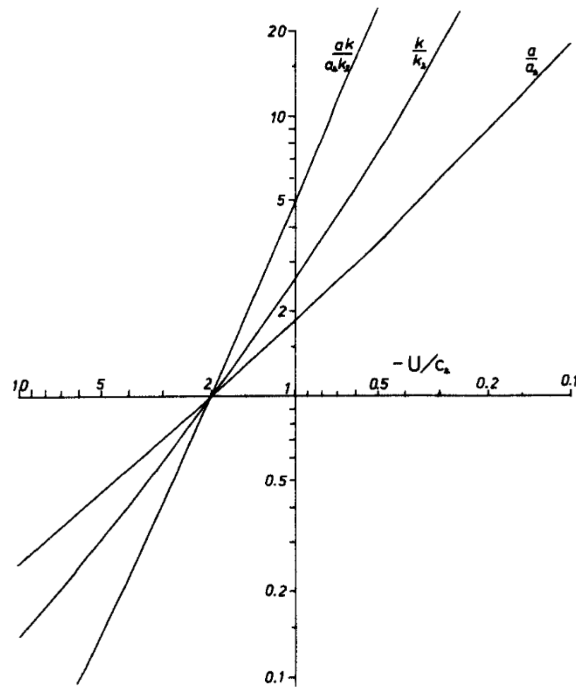


FIG. 11. The variation of amplitude, wave number, and wave steepness ak on a current $U(s)$ for negative values of ω and B. Note both ordinates are logarithmic. The suffix 2 refers to values where $U = -2c$.

figure, since for a large amplitude long wave, $-U(s)$ varies from a value greater than C in the trough to almost zero near the crest. The relatively rapid variation of steepness ak of the short wave with velocity $U(s)$ is clear.

The solutions shown in Figs. 4 and 11 can also be used when g^* differs from g. Let

$$\gamma = g^*/g_0, \tag{2.182}$$

where g_0 is the value of g^* at a point where values of the wave parameters are known. Introduce new variables

$$k^* = k/\gamma, \qquad c^* = \gamma c, \qquad a^* = a/\gamma, \qquad U^* = \gamma U. \tag{2.183}$$

Then the equations to be solved for the starred variables are those for constant $g^* = g_0$.

The simplest example with significant vertical acceleration is where the long waves are of infinitesimal amplitude with surface displacement

$$A \sin(Kx - \Omega t).$$

Then

$$U(s) \simeq -C + A\Omega \coth Kh \sin Ks, \tag{2.184}$$

$$g^*(s) \simeq g - A\Omega^2 \sin Ks. \tag{2.185}$$

Since A is small the analysis is simplified by assuming

$$U = -C + dU \qquad \text{and} \qquad g^*(s) = g + dg^*, \tag{2.186}$$

and using the differentials of Eq. (2.177) and (2.181) and the dispersion relation.

The results of this analysis, after identifying dU and dg^* with the appropriate terms in (2.184) and (2.185), and after simplifying even further by neglecting c compared with C, are

$$d\sigma/\sigma = \tfrac{1}{2}AK(\coth Kh - \tanh Kh) \sin Ks, \tag{2.187}$$

$$dk/k = AK \coth Kh \sin Ks, \tag{2.188}$$

$$da/a = \tfrac{1}{4}AK(3 \coth Kh + \tanh Kh) \sin Ks, \tag{2.189}$$

$$d(ak)/ak = \tfrac{1}{4}AK(7 \coth Kh + \tanh Kh) \sin Ks. \tag{2.190}$$

Longuet-Higgins and Stewart (1960) derive results corresponding to and agreeing with (2.188) and (2.189) in their perturbation analysis. Vincent (1975) points out that although $c < C$, it is often not small enough to be neglected. There is only a simplification of algebra gained by its neglect.

It is clear that short waves steepen as the crest of a long wave overtakes them. If the long wave has appreciable amplitude they may steepen sufficiently to break at some point on the forward facing slope of the long wave. If short waves travel in the opposite direction to the long waves, then $U(s)$ is positive (supposing that the short-wave direction is again taken to be positive). However, Eqs. (2.187)–(2.190) still hold for infinitesimal long waves since dU/U still has the same value.

It is suggested by Longuet-Higgins (1969) that the variation of steepness of short waves on longer waves may contribute to the growth or decay of the longer waves. Two mechanisms are proposed, one weak and one strong interaction. The weak effect is the viscous decay of the short waves, which is proportional to wave steepness. One result of the decay is to transfer momentum from the short waves to the "current." Since the short waves are steeper at crests than in troughs, more momentum is transferred at the crest, leading to a growth or decay according to whether the short waves are traveling with or against the longer waves. Longuet-Higgins discusses details of the momentum transfer, introducing a virtual tangential stress at the surface, and makes an estimate of its value for a typical wind-driven system.

It may account for more than 10% of the stress due to the wind at low wind speeds, and less for high winds. It is also shown that

$$\frac{1}{A}\frac{dA}{dt} = \frac{4\nu}{g^2}(ak)^2\sigma\Omega^3, \tag{2.191}$$

where ν is the kinematic viscosity; the strong dependence on the frequency of the long waves is evident, but it is possible that the effect may be significant in their amplification.

A stronger effect can be expected when the short waves steepen sufficiently to break. This gives a very direct transfer of momentum, and also, since much of the wave energy may be dissipated, more of the wave momentum may be transferred. If the wind can regenerate the short waves between the long-wave crests, this could be an efficient mechanism for generating waves with phase velocities greater than the wind speed. Longuet-Higgins' (1969) estimates for the rate of growth of the waves are consistent with this mechanism being important for certain sea states.

A contradictory result is obtained by Hasselmann (1971). The inviscid equations of motion are averaged and the interactions are considered from an Eulerian viewpoint. This represents the mass flow associated with the short waves as a surface flow occurring between the wave troughs and their crests. This results in a new kinematic boundary condition for the mean flow. At the mean free surface, $z = Z$,

$$(\partial Z/\partial t) + (\mathbf{U} \cdot \nabla_1)Z - W = -\nabla_1 \cdot \mathbf{M}/\rho, \tag{2.192}$$

where $\mathbf{M}$ is the mass flow associated with the short waves. The term on the right-hand side gives the rate at which the mean surface Z must be lowered to supply water to feed increases in the mass flow of the short waves.

This mass flow is greatest down the front face of a crest of the longer wave, effectively transferring water from the crest to the trough of the wave, and hence reducing its potential energy. Hasselmann (1971) deduces that this term is as effective in damping a wave, as the momentum transfer is at amplifying it. Using infinitesimal wave theory he analyzes the residual terms in an expression for the rate of change of the long wave and deduces that, whichever way the short waves travel, the long wave is damped. In a discussion of radar measurements of short-wave spectra he concludes that the magnitude of this damping is "of marginal significance."

The approaches in these two papers are difficult to reconcile. Clearly Hasselmann introduces an important interaction, which Longuet-Higgins was unaware of, but the momentum transfer in his inviscid model may be a poor representation. This is especially so for breaking short waves, where the vorticity generated may spread below the surface layer. Similarly, the reliance on linear theory for order of magnitude estimates may not be adequate, especially as for short waves of small amplitude there is no significant energy transfer. More work is needed on this problem, but it is reasonable to conclude that Longuet-Higgins overestimates any amplification of long waves.

A closely related subject is the generation of capillary waves near the crest of a steep gravity wave. These are interpreted as waves of the same phase velocity generated by the excess pressure at the sharply curved crest due to surface tension. Longuet-Higgins (1963) analyzes their generation by a perturbation scheme and discusses the results in the context of short waves on a large-scale flow. Crapper (1970) starts from this point of view and uses his solution for finite-amplitude capillary waves (Crapper, 1957) to calculate the variation in their steepness along the gravity wave profile. In this problem, as in most problems where surface tension is important, it is necessary to take account of the dissipation of the short-wave motion by viscosity. For this reason wave action is not conserved and it is more appropriate to use an energy equation, with radiation stress, dissipation, and an input function. For short waves of this type, if surface tension is dominant then both gravity and surface accelerations are unimportant and may be neglected. It is clear from the last few paragraphs that the dissipative effects of these capillary waves may be important and should be investigated further.

Another mechanism for generating capillary waves on the front of a gravity wave becomes evident once the properties of capillary–gravity waves are considered. These waves have a minimum value of their group velocity. We have noted that $U + c$ is always negative for short waves traveling with longer waves, in the frame of reference with the long waves stationary. For gravity waves, this implies $U + c_g$ is never zero, but this is not so for gravity–capillary waves. $U + c_g$ may be zero and there is then a stopping point. The magnitude of the velocity U at the stopping point, and hence at the crest of the long wave, must be less than the wave velocity at which $c = c_g$, which equals the minimum phase velocity. For normal gravity and clean water, this is 0.23 m sec^{-1}. These reflected capillary–gravity waves are longer than those generated at the crest. This is easily seen from graphical consideration of the equations

$$\sigma = \omega - kU,$$

$$\sigma = (gk + Tk^3/\rho)^{1/2}.$$

The reflected waves have $\omega < 0$, while those generated at the crest have $\omega = 0$ (note that $U < 0$).

A further factor influencing short waves riding on long waves on the sea is the vorticity distribution in the water due to direct wind stress and the dissipation and breaking of waves. Shemdin (1972) gives experimental measurements of velocity profiles and estimates the effect on the dispersion relation, and Banner and Phillips (1974) discuss its effect on the maximum amplitude of the shorter waves (see Section IV).

As already described, when the crest of a steep long wave catches up with short waves they increase rapidly in steepness. Dagan (1975) makes the interesting suggestion that for the highest waves, where $U(s)$ is near zero at the crests, the rapid increase of amplitude of the short waves may be interpreted as an instability of the basic flow. That is, the initiation of breaking might be described in these terms.

Two properties of the breaking process are described by this hypothesis: (i) its initiation is rapid, (ii) breaking occurs on the front face of a wave. On the other hand, breaking does not resemble an oscillatory short wave. If $U(s)$ is always greater than the minimum phase velocity, then sufficiently small-amplitude wave disturbances may pass over the wave crest, while if $U(s)$ has a lower value, there is a stopping point near the crest where waves may be reflected, and for sufficiently small initial waves, infinitesimal wave theory may be used to find the amplification. Thus the usual requirement for instability of indefinite amplification of an arbitrarily small disturbance is not met. Dagan's analysis is for a steady flow and makes no assumption about the rate of change of the short waves, in which respect it is valuable. It certainly indicates that short waves may sometimes precipitate or influence breaking.

In confirmation of this the author has a 16-mm film showing a small wave disturbance meeting a wave on the point of breaking on a beach. The larger wave breaks with two sheets of water projected forward. By running the film backward frame by frame it is clearly seen that one of the sheets of water is directly connected with the incident disturbance.

III. Small-Scale Currents

A common way of finding solutions to difficult mathematical problems is to look for parameters that may be either very large or very small and then to solve the problem for those cases by making appropriate approximations. This approach is successfully used in Section II to deal with large-scale currents. At the other extreme are current distributions with a scale much smaller than a wavelength. Some important examples are best discussed in a context covering all length scales; thus flows that vary with depth are treated in the next section, and the interaction of the flow around a ship with the waves it generates is considered in Section V. This leaves few situations of interest where much analysis may be done, and most of this section is devoted to thin shear layers.

There has been very little work on this problem, so it is of interest to note an analogous problem that has been studied more intensively. The propagation of sound through moving fluids is one such case; indeed, two-dimensional sound waves in a uniform atmosphere satisfy *exactly* the same equations and boundary conditions as infinitesimal shallow-water waves in water of constant depth. This subject is reviewed in a paper by Lighthill (1972) and the book by Goldstein (1974) gives mathematical details of some of the topics. Naturally many of the problems that arise in acoustics have little relevance to water waves and vice versa, although the mathematical methods are applicable to both fields, or at least provide a useful starting point.

When all dimensions of a current system are very small compared with those of the waves, the wave may simply be taken as giving the local mean water level and current as slowly varying functions of time. For example, this is often done in relation to the tides for small-scale coastal problems. The effect on the waves is negligible, unless there are many such small current systems. In any case, the author does not know of important or interesting examples.

More interesting currents are those which have one long length scale but which are otherwise short compared with the waves. A thin shear layer between two different, nearly uniform flows is the simplest example. A thin jet is another example. In real flows regions of strong velocity gradients only remain thin if there is some factor opposing their usual turbulent spread. However, in many cases a portion of the flow field might be well described as a thin shear layer, and any solution for that case can be of value in interpreting or predicting behavior in the problem where the current scale is of the same order as that of incident waves.

In searching for mathematical solutions to problems involving thin shear layers, it is a natural step to look at the limit as the thickness of the shear layer goes to zero. That is, to consider a flow with a vortex sheet across which the velocity is discontinuous. In practice both vortex sheets and thin shear layers are unstable flows, so that steady solutions to such problems cannot be expected to give more than a crude approximation to real situations. This is better than nothing and may be quite adequate in some circumstances.

If a vortex sheet at $y = \eta(x, z, t)$ separates two regions of flow, denoted by subscripts 1 and 2, then the boundary conditions on the vortex sheet are (i) that the pressure is continuous,

$$p_1 = p_2 , \tag{3.1}$$

and (ii) the fluid particles each side of the vortex sheet move with the sheet, that is,

$$\frac{\partial \eta}{\partial t} + u_i \frac{\partial y}{\partial x} + w_i \frac{\partial \eta}{\partial z} = v_i , \qquad \text{for} \quad i = 1, 2. \tag{3.2}$$

This latter boundary condition has been incorrectly formulated in a number of papers, both for compressible flows and for water wave problems, by the omission of the last two terms on the left-hand side of Eq. (3.2).

The simplest problem is for an undisturbed flow that consists of a plane vortex sheet with uniform flows $(U_1, 0, 0)$ and $(U_2, 0, 0)$ on its two sides. The incident wave is simplest if it is plane periodic, making an angle θ_1 with the flow direction. Solutions for the acoustic problem were given independently by Miles (1957) and Ribner (1957).

The corresponding linear shallow-water problem is quite straightforward. Matching phases on both sides of the vortex sheet gives

$$k_1 \cos \theta_1 = k_2 \cos \theta_2 , \tag{3.3}$$

$$\sec \theta_1 + F_1 = \sec \theta_2 + F_2 , \tag{3.4}$$

where θ_2 is the angle the transmitted wave makes with the flow and F_i is the Froude number $U_i/c = U_i(gh)^{-1/2}$. These show that there are *two* critical angles, and incident waves are totally reflected if

$$-1 \leq \sec \theta_1 + F_1 - F_2 \leq 1. \tag{3.5}$$

The amplitudes of waves are easily found from the linearized boundary conditions. If A_i and B_i are the amplitudes of waves propagating in the $+y$ and $-y$ directions, respectively, on the appropriate sides of the sheet, then

$$A_1 + B_1 = A_2 + B_2 , \tag{3.6}$$

$$k_2 \sin \theta_1 (A_1 - B_1) = k_1 \sin \theta_2 (A_2 - B_2). \tag{3.7}$$

If the signs of $\cos \theta_1$ and $\cos \theta_2$ differ, then the reflected and transmitted waves may be many times larger than the incident wave. This solution only occurs for

$$|F_1 - F_2| > 2 \tag{3.8}$$

and appears to be a mathematical curiosity since the amplification is much diminished if a finite shear layer is incorporated into the mathematical model (Graham and Graham, 1969).

Much further work has been done on the acoustic problem: in particular, Jones and Morgan (1972) solve for an instantaneous line source situated off the vortex sheet. After the wave produced has interacted with the vortex sheet for a finite time, an "instability wave" arises, which has an exponentially growing amplitude. Jones and Morgan (1974) use a very simple method of modeling the turbulence that must arise, and this leads to a more complete discussion of the scattered sound. The papers are also of interest for the mathematical techniques used to ensure that the solutions satisfy causality.

For waves in deep or moderately deep water, the acoustic analogy is not available. Even the simplest problem of linear plane waves on a vortex sheet has not been solved. The difficulty arises in satisfying the boundary conditions on the vortex sheet at all depths. Matching the phase of the wave in x gives

$$k_1 \cos\theta_1 = k_2 \cos\theta_2 \tag{3.9}$$

again, but matching the frequencies leads now to

$$\left[1 - \frac{(U_2 - U_1)\cos\theta_1}{c_1}\right]^2 = \frac{\cos\theta_1}{\cos\theta_2} \tag{3.10}$$

for deep water instead of (3.4), because of the different dispersion relation.

Given k_1 and θ_1, this is sufficient to determine k_2 and θ_2, and thus the range of total reflection, which again lies between two critical angles, and is given by

$$-(|\cos\theta_1|)^{1/2} \le (U_2 - U_1)\cos\theta_1/c_1 \le (|\cos\theta_1|)^{1/2}. \tag{3.11}$$

However, $k_1 \neq k_2$, except in one isolated case. Thus the variation of the wave motion with depth, that is, $\exp(k_i z)$, is different on the two sides of the vortex sheet. A solution that includes terms whose influence is confined to the neighborhood of the vortex sheet is needed.

If the velocity potentials are assumed to vary like $\exp\{i(lx - \omega t)\}$, where

$$l = k_i \cos\theta_i, \tag{3.12}$$

then the remaining problem looks deceptively simple and symmetrical. The equations to be solved are

$$(\partial^2\phi_i/\partial y^2) + (\partial^2\phi_i/\partial z^2) - l^2\phi_i = 0 \tag{3.13}$$

in the region $y \le 0$, $z \le 0$, if $i = 1$, and in $y \ge 0$, $z \le 0$, if $i = 2$. The boundary conditions on $z = 0$ are

$$\partial\phi_i/\partial z = \alpha_i^2\phi_i, \tag{3.14}$$

and those on $y = 0$ are

$$\alpha_1\phi_1 = \alpha_2\phi_2 \qquad \text{and} \qquad \frac{1}{\alpha_1}\frac{\partial\phi_1}{\alpha y} = \frac{1}{\alpha_2}\frac{\partial\phi_2}{\partial y}, \tag{3.15}$$

where

$$\alpha_i = |\omega - lU_i|g^{-1/2}. \tag{3.16}$$

Those who are mathematically inclined may like to prove the existence, or nonexistence, of solutions, or find some.

Evans (1975) has succeeded in reducing the problem to that of solving a pair of singular integral equations. Conservation of wave action is proven and approximate solutions are found. Figures 12 and 13 show the results for four angles of incidence. The reflection and transmission coefficients given are simply the ratio of the surface amplitudes in the relevant waves. It may be noted that unless θ is small the reflection is low except in the vicinity of critical angles, or when there is total reflection. When there is little reflection the transmission coefficient differs little from that for a wide shear layer for which results are given in Section II,E and shown on Fig. 12. Evans used two different approximations; both are shown in Fig. 13 and thus give an idea of the accuracy that may be expected. For

$$|U_1 - U_2| \leq c_1, \tag{3.17}$$

it appears to be quite adequate for any application.

Another solution involving vortex sheets is given by Peregrine and Smith (1975). The solution is for stationary waves on a "top-hat" jet. This type of solution is relatively simple since, on introducing a velocity potential within the jet and noting that there can be *no* motion outside it, the boundary condition (3.1) on the bounding vortex sheet reduces to

$$\partial\phi/\partial x = 0, \tag{3.18}$$

while (3.2) becomes an equation for finding its displacement. For a rectangular jet of width b and depth h, surface waves of the form

$$\zeta = a \sin(n\pi y/b) \cos lx \tag{3.19}$$

have a dispersion relation

$$gk = U^2 l^2 \tanh kh, \tag{3.20}$$

where

$$k^2 = l^2 + n^2\pi^2/b^2. \tag{3.21}$$

If the currents are taken to be as weak as the water velocities in the wave motion, then it is appropriate to make a perturbation expansion with the first approximation being a simple superposition of the two (as mentioned in Section II,C,3).

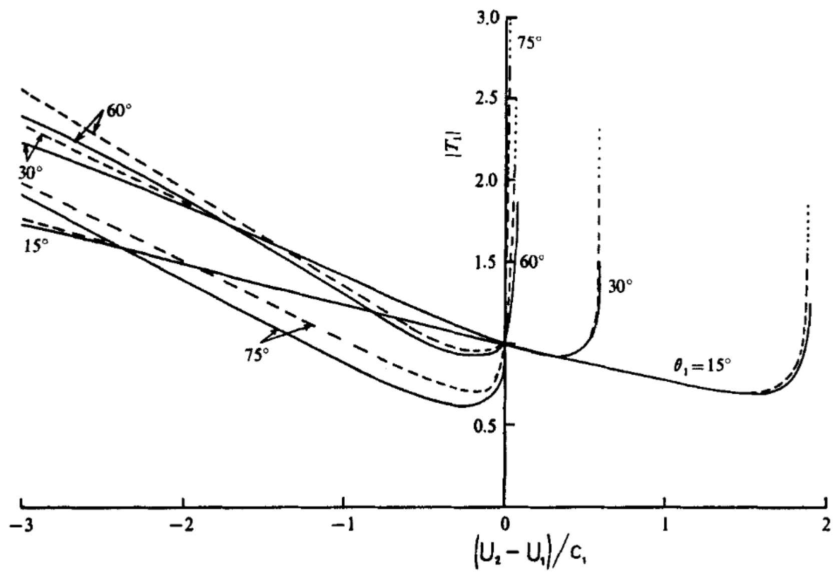


FIG. 12. The modulus $|T|$ of the transmission coefficient for waves of unit amplitude incident on a vortex sheet (solid lines) compared with the transmission coefficient for a slowly varying change of velocity (dashed lines). The angle θ_1 is the angle between the crests of the incident waves and the current and differs from the angle θ_1 used in the text. (From Evans, 1975, Fig. 1.)

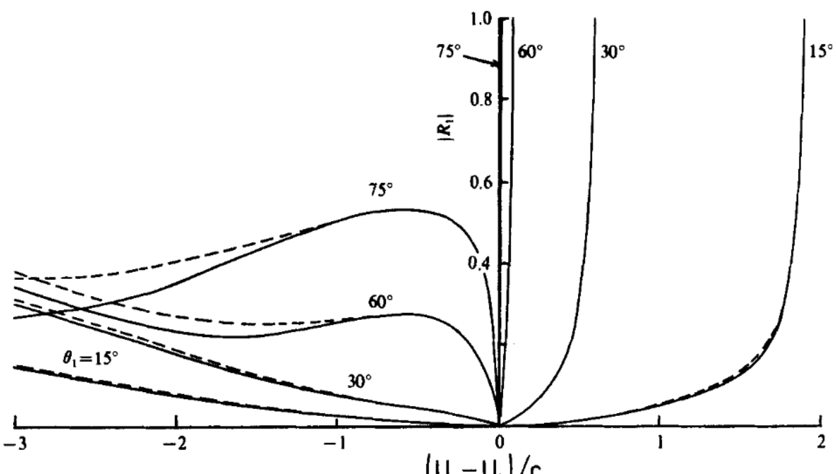


FIG. 13. The modulus of the reflection coefficient for waves of unit amplitude incident on a vortex sheet, two different approximations, solid lines and dashed lines. The angle θ_1 is the angle between the crests of the incident waves and the current and differs from the angle θ_1 used in the text. (From Evans, 1975, Fig. 2.)

IV. Currents Varying with Depth

A. Introduction

There are two major causes of steady currents that vary with depth. Wind stresses at the surface and frictional stresses acting on the bottom. Viscous stresses and turbulent Reynold's stresses transmit these to the body of the flow, setting up a mean velocity profile.

One class of such flows are those where viscosity and surface tension are important. These may be described as thin-film flows and are of great importance in chemical engineering and hence have a substantial literature, both theoretical and experimental. No more than occasional reference is made to those flows here since they are outside the scope of this paper.

For two-dimensional high-Reynolds-number flow in a stream with a free surface, the velocity profile is often taken to have the form

$$U(z) = Az^{1/7}, \qquad (4.1)$$

although measurements indicate a maximum velocity below the free surface (this may be due to surface stress from still air or to three-dimensional effects). Near a rigid bottom, $z = -h$, the velocity may be better represented by the "law of the wall" logarithmic profile

$$U(z) = (U_*/\kappa) \log[(z + h)/z_0]. \qquad (4.2)$$

Similarly, near the free surface, $z = 0$, the wind-induced current may be described by

$$U(0) - U(z) = (U_*/\kappa) \log(z/z_0). \qquad (4.3)$$

Shemdin (1972) reports measurements from a wind/wave flume and shows that they are in reasonable agreement with this formula. Wu (1975) also reports measurements from a flume that indicate a linear variation of velocity in the top few millimeters, that is, a laminar sublayer.

In many cases the major part of the velocity variation is confined to boundary layers at the surface and on the bottom. Then the wind drift will only directly affect the shortest waves and only long waves, influenced by the whole velocity profile, will be affected by the bottom boundary layer. Thus a whole range of "intermediate" length waves will be only slightly influenced by the velocity variations.

Two other causes of flows varying with depth are (i) density stratification, which can lead to internal waves and to "selective withdrawal" from a stratified reservoir, and (ii) sudden increases in depth of river beds or artificial channels with a resulting separation of the flow such that in extreme cases it may form a surface jet. Hydraulic breakwaters also take the form of a fast surface current.

In the rest of this section it is assumed that the current is in one direction only. Waves may be at an angle to the current. However, this three-dimensional problem is not discussed in most examples since if a two-dimensional solution is known then a corresponding solution at an angle θ to the current is readily found, as indicated below, following Benney (1966).

The x axis is chosen in the direction of wave propagation so that the basic flow is

$$(U(z)\cos\theta,\ U(z)\sin\theta,\ 0), \tag{4.4}$$

and the wave motion depends on x and z only. The momentum equations in the x and z directions, the continuity equation, and the boundary conditions are then exactly the same as for two-dimensional waves on the flow $U(z)\cos\theta$ in the direction of their propagation. (That is, as long as the pressure boundary condition is not rewritten by using a form of Bernoulli's equation.) This is because the motion is independent of y, and the velocity components in the y direction only occur in the y momentum equation, which thus becomes an equation for finding the y component of velocity once the rest of the problem is solved. This is true for finite-amplitude waves, but in that case can only apply for a single wave train or for two wave trains traveling in exactly opposite directions.

B. Infinitesimal Waves

1. *Equations for Variation of Wave Motion with Depth*

Taking a basic velocity field

$$\mathbf{U} = (U(z), 0, 0) \tag{4.5}$$

and the inviscid equations of motion, linearized equations for a perturbation may be written. If the depth is assumed constant with the bottom of the flow at $z = -h$, the perturbation quantities may be taken to vary like

$$f(z)\exp\{i(kx - \omega t)\} \tag{4.6}$$

without loss of generality. It is then straightforward to eliminate all but one variable, giving a second-order equation for its z variation.

If pressure is chosen, the equation is

$$p''(z) - [2U'/(U - c)]p'(z) - k^2 p(z) = 0, \tag{4.7}$$

where a prime denotes a derivative with respect to z, and $c = \omega/k$. Another convenient variable is the vertical velocity, which gives the alternative equation

$$w'' - \left(k^2 + \frac{U''}{U - c}\right)w = 0. \tag{4.8}$$

This equation is the "inviscid Orr–Sommerfeld equation" or "Rayleigh equation" of hydrodynamic stability theory. Clearly, if the velocity profile is such that it is unstable, the unstable perturbations are possible solutions as well as solutions corresponding to periodic surface waves. Stability is discussed in Section IV,D. A full discussion of this equation, in the context of the stability of flows with rigid boundaries, is given by Drazin and Howard (1966) and a shorter account may be found in Yih (1969, Ch. 9, Sect. 6).

At a rigid bottom $z = -h$, the boundary conditions for these equations are

$$p'/(U - c) = 0, \tag{4.9}$$

$$w = 0. \tag{4.10}$$

At the mean free surface $z = 0$, the linearized boundary conditions are

$$gp' = k^2(U - c)^2 p, \tag{4.11}$$

$$(U - c)^2 w' = [g + (U - c)U']w. \tag{4.12}$$

Equations (4.7) and (4.8) can only be solved explicitly for a few simple functions $U(z)$, so that it is often useful to consider composite profiles. The matching conditions at a discontinuity of velocity and/or velocity gradient are that either

$$p \quad \text{and} \quad p'/(U - c)^2 \tag{4.13}$$

or

$$w/(U - c) \quad \text{and} \quad (U - c)w' - U'w \tag{4.14}$$

be continuous. Occasionally continuity of w' has been wrongly used at discontinuities of velocity gradient instead of the second of (4.14).

For general wave numbers and frequencies, analytic solutions are only available for uniform currents and for currents depending linearly on z. Solutions for w are easily found and p is obtained from the relation

$$k^2 p = U'w - (U - c)w'. \tag{4.15}$$

Crude approximations to most velocity profiles may be made with two or more linear regions. Although analytic dispersion relations are found, even a bilinear profile leads to complicated relations that take some effort to interpret. A number of authors have used linear and bilinear velocity profiles in different circumstances. For example, Taylor (1955) finds stopping velocities for a hydraulic breakwater, and Betts (1970) studies instabilities in a flume where the flow emerges from a closed section.

For stationary waves, $c = 0$, analytic solutions may be found for a wider range of velocity profiles. Peregrine and Smith (1975) give a short table of solutions for various jetlike flows over still water, and corresponding solutions for finite depth are possible. Lighthill (1953) gives the solution for

$$U(z) = U_0(h + z)^n, \tag{4.16}$$

and Fredsøe (1974) uses the velocity profile

$$U(z) = U_1 \cos \beta(z - z_0) \tag{4.17}$$

very effectively to model stream flow over an obstacle.

There is no intrinsic difficulty in numerical integration. Fenton (1973) gives a method that is appropriate when an analytic expression is available for $U(z)$, but it may need modification if tabulated values of $U(z)$ are used since it involves $U''(z)$. Fenton presents full dispersion diagrams showing c as a function of stream velocity for a wide range of wave numbers for the simple linear profile and the one-seventh power profile (4.1). Numerical integration is also used by Shemdin (1972). For wind waves, he takes the profile (4.3) together with the corresponding velocity profile in the air. The results for the phase velocity of short waves are in agreement with his experiments.

2. *A Particular Class of Velocity Profiles*

One can give a picture of the effect of different velocity profiles on waves by considering the solutions for stationary waves on flows satisfying

$$U'' = \alpha U, \tag{4.18}$$

where α is a constant. That is,

$$U(z) = U_0 \cosh \alpha^{1/2} z + U'_0 \alpha^{-1/2} \sinh \alpha^{1/2} z, \tag{4.19}$$

where α may be positive, zero, or negative. There are three disposable constants α, U_0, and U'_0, so that this form can be used as a rough approximation to a variety of flows, and, although only stationary waves are considered, traveling waves can be included if c is supposed known.

The solution for $w(z)$ for the profile (4.19), with a rigid bottom at $z = -h$, is

$$w(z) = \sinh\{(k^2 + \alpha)^{1/2}(z + h)\}, \tag{4.20}$$

and the surface boundary conditions give the dispersion relation

$$(k^2 + \alpha)^{1/2}h \coth\{(k^2 + \alpha)^{1/2}h\} = (gh/U_0^2) + (U_0' h/U_0). \qquad (4.21)$$

If $k^2 + \alpha$ is negative, this relation still holds, noting that

$$\coth ix = -i \cot x. \qquad (4.22)$$

The corresponding dispersion relation for a uniform flow

$$kh \coth kh = gh/U_0^2 \qquad (4.23)$$

is included. Introducing

$$Z = gh/U_0^2, \qquad (4.24)$$

the inverse of a Froude number squared, we consider the two relations (4.21) and (4.23) as functions of (Z, k^2h^2). In the (Z, k^2h^2) plane the curve (4.21) is identical to (4.23) if the latter is displaced $(-U_0' h/U_0, -\alpha h^2)$, or more conveniently, if a graph of (4.23) is given, then a new origin is chosen at the point $(U_0' h/U_0, \alpha h^2)$ to give the curve (4.21).

Figure 14 shows Eq. (4.23) on the (Z, k^2h^2) plane. One advantage of this plot is that local, exponentially decaying surface disturbances are also included, in the region $k^2 < 0$. The curve has other branches, for $k^2h^2 < -\pi^2$, which do not appear in the diagram. Such disturbances are needed to describe fully flow near an obstacle or wave-generating object. Figure 15 is a complementary diagram showing velocity profiles corresponding to different choices of origin. The effects of velocity gradient and curvature in determining the position of the curve are clear. The origin is on the curve if $U(-h)$ is zero, and if the velocity profile has a zero above the bottom the corresponding origin is on the right-hand side of the dispersion curve. These cases are discussed below.

An advantage of these velocity profiles is that Eq. (4.8) has constant coefficients and it is thus straightforward, in principle, to use Fourier transforms to find solutions. Fredsøe (1974) makes use of this. He chooses an appropriate "truncated cosine" profile to model the stream flow and proceeds to calculate details of the stationary waves formed by a small oblique ridge on the bottom of a wide stream. The theory agrees well with experimental results, which Fredsøe presents for the variation of wave number with Froude number. Another example is calculated corresponding to a rounded bulge in the middle of a channel of finite width.

The results of these two calculations are shown in various ways. When compared with a uniform flow with the same *surface* velocity they exhibit a reduction in surface amplitudes. However, the paths of fluid particles at the bottom of the flow show that their transverse displacement is considerably larger (e.g., twice as much) in the shear flow than it is in irrotational flow. A

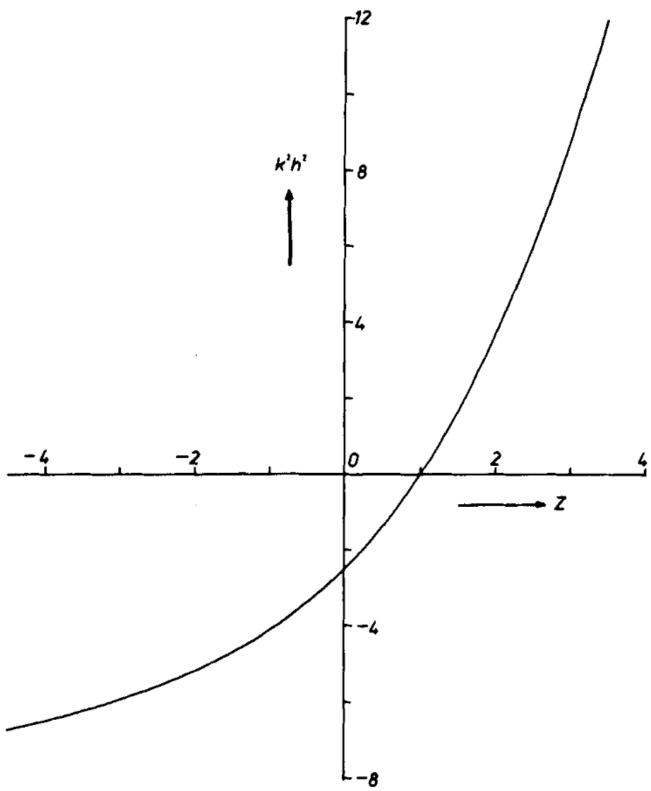


FIG. 14. The dispersion relation for stationary waves on a uniform stream. $Z = gh/U_0^2$.

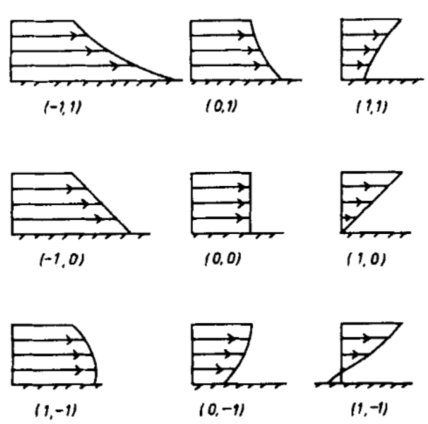


FIG. 15. The velocity profiles corresponding to the dispersion relation given by choosing the point indicated as a new origin in Fig. 14.

rough physical interpretation is given. To first order the fluid moves with the velocity appropriate to its level in the flow. Thus if its path has curvature κ, a horizontal pressure gradient of magnitude $\rho\kappa U^2$ is needed. However, the perturbation pressures are unlikely to vary in the same way as $U^2(z)$, so the curvature must vary. For the cases where perturbation pressures vary slowly compared with $U^2(z)$ the curvature must increase relatively rapidly to balance any marked reduction in $U(z)$. (Compare with boundary layer theory, where pressure is taken as constant through the layer.)

A more mathematical way of looking at this particular phenomenon is to note that Eqs. (4.7) and (4.8) have a singular point just below the bottom, in the cases studied by Fredsøe (1974).

3. *A Critical Layer in the Flow*

A singular point of Eqs. (4.7) and (4.8) occurs at $z = z_1$ if

$$U(z_1) - c = 0. \tag{4.25}$$

If $z = z_1$ is in the fluid, this means that there is a critical layer at that level. The solutions in the neighborhood of a singular point, for sufficiently differentiable $U(z)$, may be found by expanding $U(z)$ in a Taylor series in

$$Z = z - z_1.$$

As is well known from the theory of second-order differential equations, one solution is always regular and the other may be singular. In Fredsøe's example, both solutions for p and w are regular but u and v have a singular solution. If part of the singular solution is needed to satisfy the boundary condition at the bottom, u and v could be large without there being a critical layer actually in the flow.

It is instructive to look at the general solution for an oblique wave $\exp\{i(lx + my - \omega t)\}$ near a critical layer. In this case

$$lU(z_1) - \omega = 0, \tag{4.26}$$

or

$$U(z_1)\cos\theta = c. \tag{4.27}$$

The results are

$$p = B[1 - \tfrac{1}{2}k^2Z^2 - 2k^2(U_1''/U_1')Z^3 \log|Z|] + AZ^3 + O(Z^4 \log|Z|), \tag{4.28}$$

$$(u, v, w) = \frac{Bm}{l^2 U_1' Z}(m, -l, 0) + \frac{2Bk^2 U_1''}{(U_1')^2} \log|Z| \; (1, 0, 0)$$
$$+ \frac{BU_1''}{l^2 (U_1')^2}(3l^2 + m^2, ml, -ik^2 U_1'/U_1'')$$
$$- \frac{3A}{l^2 U_1'}(1, 0, 0) + O(Z \log|Z|), \tag{4.29}$$

where A, B are constants multiplying the regular and singular solutions, respectively, and $k^2 = l^2 + m^2$.

The most striking aspect of this result is that the most singular behavior is in the perturbation velocity in a direction *perpendicular* to the wave number vector of the wave train. This Z^{-1} term goes to zero for waves traveling against the current since its variation with θ, the angle between $\mathbf{k}$ and $\mathbf{U}$, is $\sin\theta \sec^2\theta$. This becomes very large for θ near $\pi/2$, but the possibility of a critical layer then is remote because condition (4.27) becomes difficult to satisfy at a depth where the effects of the wave motion are significant.

The singular solution cannot be used directly as a description of wave motion. At a critical layer other properties of the flow, neglected in the present analysis, must be introduced in order to find a physically sensible description. This aspect of critical layers is extensively studied in the theory of hydrodynamic stability. The usual method of proceeding is to include the effects of viscosity, which are used to get a solution valid in the neighborhood of the critical layer. This may be matched with an inviscid solution each side of a layer. This is relatively straightforward for unstable, growing modes of which there are usually only a finite number. The inclusion of viscosity also introduces a set of damped modes. For the water wave problem, Craik (1968) presents an analysis for resonant interactions among a triad of waves on flow with a uniform shear, and Velthuizen and van Wijngaarden (1969) consider long waves in a channel and attempt to find their rate of decay. Velthuizen and van Wijngaarden are concerned about the problem of upstream propagation against fast flows (see Section IV,B,5) so they assume a critical layer for very long waves even though there is a solution without a critical layer. However, a full discussion of solutions with critical layers should take into account how the waves may be generated, and a new proposal is presented below.

For high-Reynolds-number flows it may be more appropriate to include nonlinear effects or effects due to the turbulence in the flow to find a local solution for the critical layer. No such applications have been made to this field.

For a realistic class of flows satisfying

$$U'(0) < 0, \qquad U''(z) \geq 0, \qquad U'(z) \quad \text{finite}, \tag{4.30}$$

in $-h \leq z \leq 0$, Yih (1972) shows that if there is a critical layer at $z = z_1$, then c must be real and the boundary conditions are not consistent with a solution for which $w(z_1)$ is nonzero. He incorrectly excludes the case $w(z_1) = 0$. By multiplying Eq. (4.8) by the complex conjugate function $w^*(z)$ and integrating from $-h$ to z_1, he finds

$$\int_{-h}^{z_1} (|w'|^2 + k^2 |w|^2)\, dz + \int_{-h}^{z_1} \frac{U''}{U - c} |w|^2\, dz = 0, \tag{4.31}$$

after putting $w(z_1) = w(-h) = 0$. Since conditions (4.30) imply $U''/(U - c)$ is positive in $-h \leq z < z_1$, the only possible value for $w(z)$ in that interval is zero. There may be a discontinuity in $w'(z)$ at a singular point, and thus a solution regular for $z \geq z_1$ and zero for $z \leq z_1$ is possible.† Direct examination of the equations of motion shows that this type of solution satisfies them, and it will be called a "surface layer solution." Such a solution is also possible for other flows; they do not need to satisfy conditions (4.30), so that other solutions may also be possible in some cases.

Where more than one solution is possible, the relevant one in any circumstance might be determined by solving an initial value problem. It seems reasonable that if the waves are generated by surface disturbances or by disturbances above the critical layer then the surface layer solution is appropriate; but if the wave generation is by a disturbance extending below the critical layer, other possible solutions may be expected to be relevant.

It is desirable to ascertain when the surface layer dispersion relation differs significantly from the "conventional" solution. As usual, the only case that is simple to investigate analytically is the linear profile, for example,

$$U(z) = U_0(1 + z/h). \tag{4.32}$$

† When this result was communicated to Professor Yih, he agreed that it is possible for the differential equation (4.8) to have a solution with a discontinuous w' at $z = z_1$, a possibility that had simply escaped his attention. He notes, however, that for $k = 0$ the solution regular at $z = z_1$ is

$$w = U - c,$$

and this cannot possibly satisfy the free-surface condition. Thus for long waves Yih's conclusion still stands. How large k^2 has to be in order to have a solution with discontinuous w' can be decided by following the development in Yih's paper (1972, pp. 214–216) for the case $U''(z_1) = 0$. The condition $U''(z_1) = 0$ can now be removed since we admit a solution with a discontinuous w', and an estimate of k for the longest possible waves may be made using the long-wave approximation given in Section IV,B,4.

This has the "conventional" dispersion relation

$$k \coth kh = g/(U_0 - c)^2 + U_0/h(U_0 - c). \tag{4.33}$$

If there is a critical layer at a depth h_1,

$$h_1 = (U_0 - c)h/U_0, \tag{4.34}$$

and the surface layer dispersion relation is

$$k \coth kh_1 = g/(U_0 - c)^2 + U_0/h(U_0 - c). \tag{4.35}$$

The two dispersion relations can only differ appreciably if kh_1 is sufficiently small for tanh kh_1 to be noticeably less than one, say $kh_1 < 2$. The dispersion relation (4.35) can be rewritten in this particular case as

$$kh_1 \coth kh_1 = 1 + gh/(U_0 - c)U_0, \tag{4.36}$$

the right-hand side of which has a minimum value $1 + gh/U_0^2$ for $0 \le c \le U_0$. For kh_1 to be less than 2, the left-hand side of (4.36) must also be less than 2. Thus the surface layer dispersion relation differs significantly from the conventional one only if

$$gh/U_0^2 < 1. \tag{4.37}$$

But this condition is appropriate for the case $c = 0$, where in fact they agree, so in practice the condition is

$$U_0^2 \gg gh, \tag{4.38}$$

which implies that the shear must be quite large. The two dispersion relations are plotted, in two different ways in Fig. 16 for the case

$$U_0^2 = 4gh_0. \tag{4.39}$$

These results are of academic interest only since such high-speed flows develop finite-amplitude waves as instabilities of greater practical importance (see Section IV,D).

For short waves, for which the depth of the flow is not significant, it is straightforward to show that the surface layer dispersion relation is only significantly different if

$$U_0' > \tfrac{1}{2}\sigma, \tag{4.40}$$

where σ is the frequency of the waves relative to the surface water, that is, $\omega - kU_0$. Again, this is a strong shear.

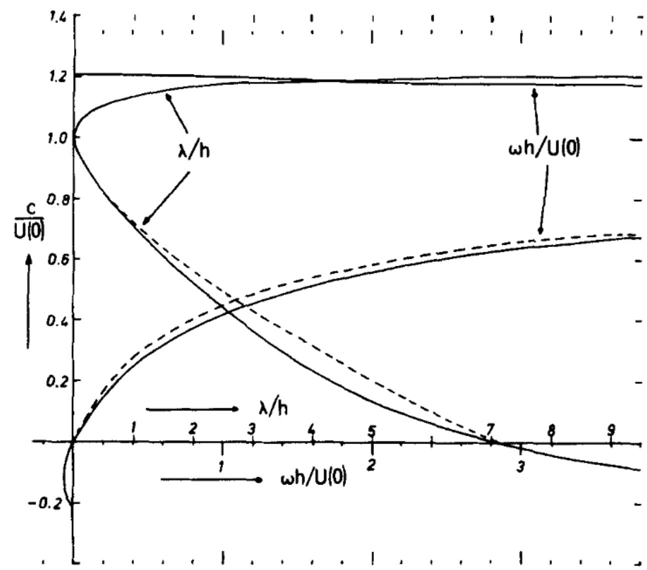


FIG. 16. Dispersion curves for the velocity profile $U(z) = 2(gh)^{1/2}(1 + z/h)$. The dashed line is for the surface layer solution.

4. *Approximate Solutions*

Approximations may be made for long waves and for short waves. For long waves, $kh \ll 1$ and an appropriate way to write Eq. (4.7) is

$$[p'/(U - c)^2]' = k^2 p/(U - c)^2, \tag{4.41}$$

which may be integrated twice to give the integral equation

$$p(z) = A + B \int_{-h}^{z} (U_1 - c)^2 \, dz_1 + k^2 \int_{-h}^{z} \int_{-h}^{z_2} \frac{(U_2 - c)^2}{(U_1 - c)^2} p(z_1) \, dz_1 \, dz_2 , \tag{4.42}$$

in which the abbreviation

$$U_m = U(z_m), \tag{4.43}$$

is used, and A and B are constants to be determined by boundary conditions. It is now easy to find successive approximations to $p(z)$ as a power series in k.

For flow over a rigid bottom, $B = 0$, and, setting $A = 1$ without loss of generality,

$$p(z) = 1 + k^2 \int_{-h}^{z} \int_{-h}^{z_2} \frac{(U_2 - c)^2}{(U_1 - c)^2}\, dz_1\, dz_2 + k^4 \int_{-h}^{z} \int_{-h}^{z_4} \int_{-h}^{z_3} \int_{-h}^{z_2} \frac{(U_4 - c)^2 (U_2 - c)}{(U_3 - c)^2 (U_1 - c)}\, dz_1\, dz_2\, dz_3\, dz_4 + O(k^6 h^6), \tag{4.44}$$

with a dispersion relation

$$g \int_{-h}^{0} \frac{dz_1}{(U_1 - c)^2} + gk^2 \int_{-h}^{0} \int_{-h}^{z_3} \int_{-h}^{z_2} \frac{(U_2 - c)^2}{(U_3 - c)^2 (U_1 - c)^2}\, dz_1\, dz_2\, dz_3 = 1 + k^2 \int_{-h}^{0} \int_{-h}^{z_2} \frac{(U_2 - c)^2}{(U_1 - c)^2}\, dz_1\, dz_2 + O(k^4 h^4). \tag{4.45}$$

This result is given by Thompson (1949), but the first approximation

$$g \int_{-h}^{0} dz_1/(U_1 - c)^2 = 1 \tag{4.46}$$

is better known from Burns' (1953) paper.

The same approach can be used for problems where the flow is uniform except for a thin layer, e.g., a boundary layer at the bottom or at the free surface, or to the surface layer solution when that layer is thin. For example, consider waves on still, deep water with a thin wind-driven boundary layer of thickness h. The pressure perturbation $p(z)$ must vary as $\exp(kz)$ below the layer, and if U is effectively zero at $z = -h$, matching p and p' with solution (4.42) leads to

$$c^2 A = kB, \tag{4.47}$$

and to the approximate dispersion relation

$$\omega^2 \left[1 + \frac{k}{c^2} \int_{-h}^{0} (U_1 - c)^2\, dz_1\right] = gk\left[1 + kc^2 \int_{-h}^{0} \frac{dz_1}{(U_1 - c)^2} + O(k^2 h^2)\right]. \tag{4.48}$$

Not one of the dispersion relations (4.45), (4.46), and (4.48) is easy to use or interpret. Perhaps the simplest is the first approximation to stationary waves on a surface jet. This is obtained by putting $\omega = 0$ in (4.48) and leads to the result

$$g - k^2 \int_{-h}^{0} U^2(z)\, dz = 0, \tag{4.49}$$

given by Peregrine and Smith (1975). The integral is proportional to the momentum flow in the surface jet. A thin sheet of momentum flow at the surface of a fluid acts rather like a negative surface tension. Compare Eq. (4.49) with

$$g + (Tk^2/\rho) = kc^2, \tag{4.50}$$

the deep-water dispersion relation, when $c = 0$ (but also see "surface shear waves" in Section IV,C).

At the other extreme, when waves are short compared with the current variations, wave properties are determined by the flow close to the surface. One can either use a WKB approximation for $w(z)$ as Dalrymple (1973, Appendix 1) does, or expand systematically in inverse powers of

$$k_0 = g/(U_0 - c)^2, \tag{4.51}$$

as is done by Peregrine and Smith [1975, Eq. (48)]. The first three terms of the dispersion relation are

$$k = \frac{g}{(U_0 - c)^2} + \frac{U_0'}{(U_0 - c)} - \frac{(U_0 - c)U_0''}{2g}, \tag{4.52}$$

where a zero subscript indicates that the function is evaluated at the surface. Further approximations involve higher derivatives of $U(z)$ that would be difficult to evaluate from measurements of a real flow.

Some idea of the accuracy of these approximations may be obtained from Fig. 17 which shows the dispersion relation for stationary waves on a deep flow of the form

$$U(z) = U_0 e^{\alpha z},$$

with k plotted against U_0, using appropriate dimensionless variables.

A few bounds for c are available. Thompson (1949) proves

$$U_{\min} - (gh)^{1/2} \le c \le U_{\max} + (gh)^{1/2}, \tag{4.53}$$

and that when U is a monotonic and nondecreasing function of height above the bed,

$$c \le U_{\max} + (g/k)^{1/2}. \tag{4.54}$$

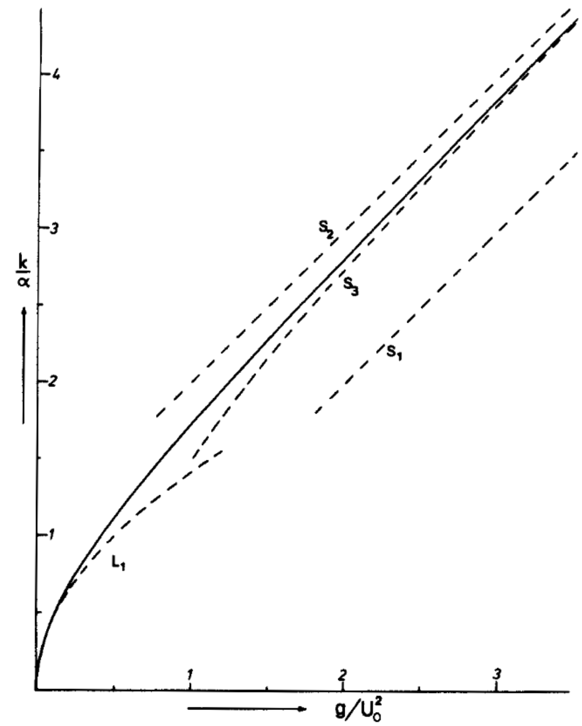


FIG. 17. Dispersion relation for two-dimensional stationary waves on the velocity profile $U(z) = U_0 \exp \alpha z$, together with approximations. S_1, S_2, and S_3 are successive short-wave approximations and L_1 is the first long-wave approximation. (From Peregrine and Smith, 1975, Fig. 4.)

With similar conditions,

$$U' \geq 0 \quad \text{and finite}, \qquad U'' \leq 0, \tag{4.55}$$

Yih (1972) extends a result of Burns (1953) to prove that there is one solution with

$$c \leq U(-h), \tag{4.56}$$

and another with

$$c \geq U(0). \tag{4.57}$$

It is worth noting that there is no simple equivalent of the linear long-wave equations for irrotational flow:

$$\frac{\partial u}{\partial t} + g\frac{\partial \zeta}{\partial x} = 0, \qquad \frac{\partial \zeta}{\partial t} + h\frac{\partial u}{\partial x} = 0. \tag{4.58}$$

There is only the result (4.46) for the long-wave velocity.

5. *Upstream Propagation*

Conditions (4.55) are such that many profiles $U(z)$ that may be chosen to represent stream flow would satisfy them. If $U(-h)$ is zero, condition (4.56) indicates that that solution corresponds to upstream propagation of waves, regardless of how large the surface or mean velocities may be. Benjamin (1962, p. 108) suggests that the solution (4.56) may not be physically realizable when the mean velocity $\bar{U}$ is much greater than $(gh)^{1/2}$. He argues that the relatively high velocities near the bed due to the wave in such a solution severely limit the amplitude of the wave if separation of the boundary layer is not to occur. Yih (1972) discusses this further, confirming the high perturbation velocities near the bed, and arguing against a conjecture of Benjamin's that the maximum velocity of propagation upstream should be of the order $-(gh)^{1/2} + \bar{U}$.

There are several facets to this problem. There is no doubt of the existence of the mathematical solution corresponding to (4.56) with conditions (4.55) for any value of $\bar{U}$. The conditions (4.55) certainly apply to a real flow if the Reynolds number is low enough for it to be laminar. However, in that case it is less realistic to omit viscosity in the analysis. If viscosity is included these waves do not occur, as is shown by a stability analysis [see Benjamin (1957) or Yih (1969, Sect. 9.9) for further results].

For a turbulent high-Reynolds-number flow, many model profiles $U(z)$ would give either a nonzero velocity at the bottom or an infinite velocity gradient $U'(-h)$ as in the frequently used one-seventh power profile (4.1). In the latter case, Lighthill (1953) shows there is no upstream propagation for

$$\bar{U} > 1.0353(gh)^{1/2}. \tag{4.59}$$

There is also the problem of generation and detection of such waves. For example, using the linear velocity profile (4.34) a rough calculation shows that if a low-frequency oscillating surface pressure is applied over an appropriate length of the surface, the two long-wave modes are generated with amplitudes inversely proportional to their phase velocities. This means that for a high-Froude-number flow, the controversial upstream propagating mode would have a substantially smaller amplitude than the waves propagating downstream. Its group velocity upstream would also be small, so that it would suffer appreciable damping, by neglected effects, before it got clear of the generating area.

In summary, these particular upstream propagating waves appear to be a mathematical solution with little physical relevance. There is an exception, when the flow separates from the bed. This is discussed further in Sections IV,C and D.

6. *Group Velocity*

In many applications, the most important wave parameter is the group velocity; for example, it is needed to find the stopping velocity in a hydraulic breakwater. Once again, the only simple case is the linear velocity profile

$$U(z) = U_0 + zU'_0 . \tag{4.60}$$

The group velocity c_g is given by

$$c_g - U_0 = \frac{(c - U_0)[\beta(kh) + \{\frac{1}{2} - \beta(kh)\}(c - U_0)U'_0/g]}{1 - \frac{1}{2}(c - U_0)U'_0/g}, \tag{4.61}$$

in which $\beta(kh)$ is the ratio c_g/c for waves of the same wave number in still water of depth h. The denominator of the right-hand side of (4.61) cannot be less than $\frac{1}{2}$ since the maximum value of $(c - U_0)U'_0/g$ is 1, which it attains at $k = 0$.

From Eq. (4.61) one may see that $(c_g - U_0)/(c - U_0)$ behaves rather like $\beta(kh)$ but "skewed" in the direction one would expect from the underlying shear. Inspection of Fig. 16 shows that the surface layer solution has similar properties, except at $c = 0$, a point that may merit further attention.

For most applications, numerical solutions need to be found; even when analytic solutions are found it may be more convenient to determine c_g graphically or numerically, e.g., Taylor (1955) finds the stopping velocities of a sectionally linear profile graphically.

C. Finite-Amplitude Waves

One familiar method of finding finite-amplitude wave solutions becomes relatively inappropriate when the basic flow varies with depth. This is the method of expanding the free-surface boundary condition in a Taylor series about the mean level. If this approach is adopted, the mean flow $U(z)$ must also be expanded in a Taylor series. While this may be sensible for a flow chosen for its mathematical convenience, such as a linear profile with constant vorticity, it is quite inappropriate if actual velocity measurements are used, since even second derivatives may be quite uncertain.

A number of transformations of the equations of motion for steady flow enable this problem to be avoided, at least for steady periodic waves. A few transformations are now given, followed by some solutions.

1. *Transformations of the Equations*

For steady flow in two dimensions, the introduction of a stream function ψ leads to the expression $-\nabla^2\psi$ for vorticity and to the equation

$$\nabla^2\psi = f(\psi) \tag{4.62}$$

to express the fact that for inviscid flows vorticity is constant along streamlines (Batchelor, 1967, Sect. 7.4). Transformations of coordinates that effectively replace z with ψ thus have two desirable properties. The free surface becomes a fixed boundary, ψ = const, and the basic distribution of vorticity is explicitly stated.

The most obvious transformation is the direct one,

$$\mathbf{u}(x, z) = \mathbf{u}(x, \psi), \tag{4.63}$$

a von Mises transformation. The continuity equation and Eq. (4.62) become

$$\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial \psi} - u\frac{\partial w}{\partial \psi} = 0, \tag{4.64}$$

$$\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial \psi} + u\frac{\partial u}{\partial \psi} = f(\psi), \tag{4.65}$$

respectively. These are two equations for the two components of the total velocity in a reference frame moving with the wave. Gouyon (1958) and Moiseev (1960) have used these equations for existence proofs in the case where the variation in the basic flow is small compared with the wave velocity, so that to a first approximation they are additive.

A less direct approach is to use the height z of a streamline as an independent variable, that is,

$$z = z(x, \psi). \tag{4.66}$$

The total velocity components are then

$$u = 1/z_\psi \qquad \text{and} \qquad w = z_x/z_\psi\,, \tag{4.67}$$

where subscripts are used to denote partial derivatives. The vorticity equation (4.62) becomes

$$z_{xx}z_\psi^2 - 2z_{x\psi}z_x z_\psi + z_{\psi\psi}(1 + z_x^2) = z_\psi^3 f(\psi). \tag{4.68}$$

Dubreil-Jacotin (1934) uses this equation for an existence proof. Dalrymple (1973) gives a finite-difference approximation to Eq. (4.68) and presents sample results of finite-amplitude waves on a linear shear and on a one-seventh power profile.

Benjamin (1962) takes this approach a step further in his derivation of a solitary-wave solution. He introduces a new height variable s, equal to the value of z in the undisturbed flow. Thus the undisturbed flow is given by

$$\psi = \Psi(s), \tag{4.69}$$

where

$$U(s) - c = d\Psi/ds. \tag{4.70}$$

The vorticity equation now becomes

$$(U(s) - c)\{z_{xx}z_s^2 - 2z_{xs}z_xz_s + z_{ss}(1 + z_x^2)\} + U'(s)\{z_s^3 - z_s(1 + z_x^2)\} = 0. \tag{4.71}$$

An advantage of this equation is that $U(s)$ appears explicitly. Note that it is a nontrivial matter to find $f(\psi)$ to substitute in Eqs. (4.62), (4.65), and (4.68) for most velocity profiles, since it is given by

$$f(\psi) = \Psi''(s), \qquad \psi = \Psi(s). \tag{4.72}$$

2. *Solutions*

The earliest finite-amplitude wave solution is Gerstner's (1802, in Lamb, 1932, Sect. 251) and it has a vorticity distribution. As Lamb shows, following Stokes, the uniform flow corresponding to that vorticity distribution is

$$U(z) = -ce^{2kb}, \tag{4.73}$$

where

$$k(z - z_0) = kb - \tfrac{1}{2}e^{2kb}, \qquad c^2 = g/k, \qquad kz_0 = \tfrac{1}{2}a^2k^2 - \ln ak.$$

The flow is in the opposite direction to the waves' propagation and for the highest wave the vorticity is singular at the free surface. This solution is unlikely to be relevant to waves on real flows.

As might be expected from the difficulty of finding solutions for infinitesimal waves for most velocity profiles, the only analytic solution corresponding to the Stokes wave for irrotational flow is for flow with uniform vorticity. Tsao (1959) gives a third-order approximation for arbitrary depth. The algebraic complexity of the solution is somewhat daunting, despite the fact that for uniform vorticity it is possible to introduce a velocity potential for the wave motion.

For the linear profile

$$U = bz, \tag{4.74}$$

it is relatively simple to show that for deep-water waves, the wave motion is given by a velocity potential

$$ace^{kz} \sin k(x - ct) + \tfrac{1}{4}a^2b(3 + S)e^{2kz} \sin 2k(x - ct) + O(a^3k^3), \tag{4.75}$$

with a surface elevation

$$\zeta = a \cos k(x - ct) + \tfrac{1}{2}a^2k(1 + 2S + \tfrac{1}{2}S^2) \cos 2k(x - ct) + O(a^2), \tag{4.76}$$

in which

$$kc^2 + bc - g = 0 \tag{4.77}$$

$$S = b/ck = b/\sigma. \tag{4.78}$$

The constant in Bernoulli's equation is increased by

$$C_2 = \tfrac{1}{2}a^2b(ck + \tfrac{1}{2}b), \tag{4.79}$$

and this may be interpreted as either a change of level of the free surface by $C_2/(g - bc)$ or an additional uniform velocity $-C_2/c$. Tsao (1959), Eq. (2.19), is not in agreement with the result (4.79).

From the wave elevation (4.76) it is easily seen that the usual asymmetry between crest and trough increases as the shear increases for waves traveling in the $+x$ direction. That is, if the maximum current is at the surface, waves traveling in that direction have sharper crests than corresponding irrotational waves. Conversely, waves traveling in the opposite direction are more nearly symmetrical about the mean level. The wave profile is sinusoidal to second order when

$$S = -2 + \sqrt{2}. \tag{4.80}$$

We may see that

$$S \geq -1 \tag{4.81}$$

by rewriting the dispersion relation (4.77) in terms of S,

$$c^2(1 + S) = g/k. \tag{4.82}$$

However, for c negative a surface layer solution should be found, which may modify the results.

Numerical techniques to solve the finite-amplitude problem for steady waves have been developed by Dalrymple (1973, 1974). He extends Dean's stream function method (Dean, 1965), which essentially is a double Fourier expansion of the stream function. A considerable number of results are presented for waves of flows with linear velocity profiles. The results include large-amplitude waves. Dalrymple (1973) also presents some results from a finite-difference approximation to Eq. (4.68) for other velocity profiles. These methods appear to be an effective approach to solving specific problems, and by comparison with irrotational waves, the effects of vorticity may be better understood.

Solitary-wave solutions may be found for any velocity profile without a critical layer. The solution is given by Benjamin (1962) and has been derived by several other authors since then. The corresponding Korteweg de Vries equation is given by Benney (1966) [which appears to have an error in Eq. (54)] and by Freeman and Johnson (1970). The equation is

$$2I_3\left(\frac{\partial \zeta}{\partial t} + c_0\frac{\partial \zeta}{\partial x}\right) - 3gI_4\frac{\partial \zeta}{\partial x} = J\frac{\partial^3 \zeta}{\partial x^3}, \tag{4.83}$$

where

$$I_n = \int_{-h}^{0} \frac{dz}{(U - c_0)^n}, \tag{4.84}$$

$$J = \int_{-h}^{0}\int_{z_3}^{0}\int_{-h}^{z_2} \frac{(U_2 - c_0)^2\, dz_1\, dz_2\, dz_3}{(U_3 - c_0)^2(U_1 - c_0)^2}, \tag{4.85}$$

and c_0 is the linear long-wave velocity given by

$$gI_2 = 1. \tag{4.86}$$

Note that the I_n are negative for n odd.

The solitary-wave solution is

$$\zeta = a\,\mathrm{sech}^2\,\beta(x - c_1 t), \tag{4.87}$$

in which

$$c_1 = c_0 - aI_4 g/2I_3, \tag{4.88}$$

$$\beta^2 = aI_4 g/4J. \tag{4.89}$$

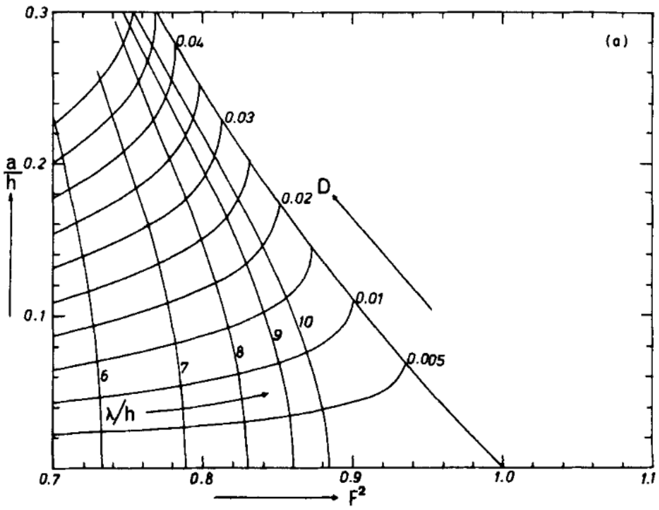
These reduce to the usual irrotational results for water at rest far from the wave,

$$c_1 = (gh)^{1/2}(1 + \tfrac{1}{2}a/h), \qquad \beta^2 = 3a/4h^2. \tag{4.90}$$

Benjamin (1962) discusses the results for waves propagating on a stream and deduces that the effect of the vorticity is small unless the Froude number of the flow is near one. In that case the waves most commonly met are stationary waves. Strictly in such a case there is a critical layer at the bottom of the flow, but if the one-seventh power velocity distribution (4.1) is used to model the flow the analysis is not affected since all the integrals converge. A comparison, of stationary waves on such a flow, with the corresponding irrotational flow, is given in Fig. 18.

The example calculated corresponds to the waves generated by a small obstacle, stationary in a stream with Froude number

$$F = \bar{U}(gh)^{-1/2}. \tag{4.91}$$



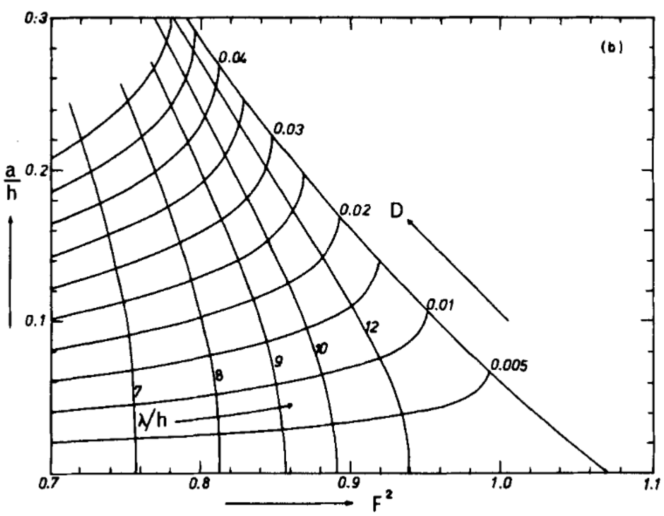


FIG. 18. Stationary waves on a stream caused by a reduction $\rho \bar{U}^2 h D^2$ in the momentum flow. (a) Uniform stream of velocity $\bar{U}$. (b) Stream with velocity profile. $(8/7)\bar{U}(z + h)^{1/7} h^{-1/7}$.

It is supposed that the momentum flow

$$S = \rho \int_{-h}^{\zeta} (p + \rho u^2)\, dz \tag{4.92}$$

is reduced by a small amount

$$\rho \bar{U}^2 h D^2, \tag{4.93}$$

which corresponds to the force on the obstacle. The energy of the flow is assumed to be unchanged. [See Benjamin and Lighthill (1954) for a full discussion in the case of uniform flows, and Fenton (1973) for an application to a linear profile.] The right-hand boundary in the figures corresponds to the solitary-wave solution. At first sight the slope of this curve is surprising, since the larger the wave the faster it travels. However, one result of the loss of momentum flow is to reduce the mean level of the stream, so there is no inconsistency in a larger wave being on a slower stream. The most noticeable difference between Figs. 18a and b is the increase with vorticity of the area of the (a, F^2) plane in which waves may occur. Essentially this is due to the increased speed of the solitary wave, which is noted by Benjamin (1962). Other differences are quantitative rather than qualitative. The biggest of these is an increase in wavelength for the stream with vorticity when $F^2 < 0.85$, but the theory is less appropriate there.

For finite-amplitude shallow-water waves, the position is similar to that for linear long waves. Benney (1974) shows that there is no pair of "simple shallow water equations to characterize long waves in a general flow." On the other hand, if attention is focused on waves propagating in one direction only, some progress has been made by Blythe *et al.* (1972). By looking for an equation of the form

$$(\partial\zeta/\partial t) + c\,\partial\zeta/\partial x = 0, \tag{4.94}$$

where c is a function of ζ, equations leading to a simple-wave solution are found for flows without a critical layer. An example is given for the case where the flow has uniform vorticity, but no guidance is given for other less simple flows.

A rather specialized class of finite-amplitude waves has been described by Peregrine (1974) and named "surface shear waves." The basic flow configuration is a sheet of rapidly moving water traveling over water at rest or nearly so. Stationary waves may form on such a flow. Below weirs or sluices they may have an amplitude much greater than the initial thickness of the surface sheet. The wide range of conditions in which this form of wave occurs is shown by Moore and Morgan (1959), who call it a "wave hydraulic jump."

A very simple theory is possible when the Froude number

$$(M/\rho g h^2)^{1/2} \tag{4.95}$$

is large. Here h is the thickness of the jet and M its momentum flow,

$$\int_{-h}^{0} \rho U^2(z)\,dz. \tag{4.96}$$

To a first approximation the surface flow is deflected only by the pressure difference across it. That is the difference between atmospheric pressure and the approximately hydrostatic pressure in the slow-moving water beneath. The appropriate equation for the surface elevation is

$$M\kappa = \rho g \zeta, \tag{4.97}$$

where κ is the curvature of the surface. This is also the "elastica" equation for the bending of a thin sheet of elastic material. Thus the shape of the waves is easily reproduced by bending a sheet of paper.

The steepness of the waves is not limited in theory, and in practice it is easy to produce them with slopes greater than 30°, the maximum for Stokes waves. The crests and troughs are rounded and symmetrical.

Peregrine (1974) also draws attention to superficially similar waves that occur on beaches in the backwash from surf, described in more detail in the next section. If their structure is also similar, then they are formed by the high-velocity backwash separating from the beach and riding up over a separation bubble. This hypothesis is supported by experiments with the wave hydraulic jump, where by raising backwater levels, it can be formed by separation of flow from the plane spillway of a weir.

3. *Highest Waves*

For irrotational waves, Stokes (Lamb, 1932, Sect. 250) showed that the highest waves in steady motion have a 120° corner at their crest. The same method of local analysis can be applied to waves on a rotational flow, and Miche (1944, pp. 386–406) shows that the result is unchanged. Miche proceeds further and shows that vorticity affects the curvature each side of the crest. In particular, vorticity b gives a free surface

$$\theta = \pm\frac{\pi}{3} + \frac{b}{3}\left(\frac{r}{g}\right)^{1/2} + \cdots, \tag{4.98}$$

where θ is measured from the downward vertical. Grant (1973) shows that for irrotational waves the next term in an expansion like (4.98) has a (probably) transcendental power of r approximately $r^{1.2}$.

Delachenal (1973) also considers this problem but assumes that the vorticity has the form

$$Ar^{-1/2}f(\theta), \tag{4.99}$$

which is singular at the crest. Thus his solution is in the same unrealistic class as Gerstner's highest wave.

A different aspect of highest waves is the maximum amplitude that may be attained. If the phase velocity of the wave in question is known, then the use of Bernoulli's theorem in a frame of reference moving with the wave gives the maximum amplitude corresponding to a stagnation point at the crest. Banner and Phillips (1974) draw attention to the effect a surface drift due to wind has in this context. If the surface drift velocity has magnitude q, then the maximum amplitude of a wave of phase velocity c relative to deep water in the same direction as q is

$$\frac{c^2}{2g}\left(1 - \frac{q}{c}\right)^2. \tag{4.100}$$

Note that for an irrotational wave of given wavelength c only varies by 20% as the amplitude increases to its maximum. It is unlikely that a thin shear layer can cause a much larger variation. Thus if q/c is near 1, the maximum wave can be expected to have quite a small amplitude, and any small amplitude approximation is likely to be very limited in its applicability. As Banner and Phillips point out, this is likely to be the case for the shorter waves in a wind-driven sea.

Phillips and Banner (1974) investigate the effect of large waves on a surface drift layer. The water motion in the large wave causes the surface drift to vary, with its maximum velocity at the crests of the long wave. As discussed in Section II,F, short waves become shorter near long-wave crests, and thus their phase velocity relative to the water below the surface layer decreases, just when the velocity of that layer increases. When $q = c$ they cannot propagate, and however small their amplitude, it seems that they must break. Phillips and Banner estimate these effects using linear wave theory. If the long wave has appreciable steepness, then the amplification of the surface drift at the crest of the wave is substantial. Thus the interaction of long surface waves with a surface drift layer may have appreciable effects in suppressing shorter waves. The proportionate reduction in wave energy is estimated for the shorter waves and is found to be in surprisingly good agreement with experimental measurements.

For irrotational waves, Longuet-Higgins and Fenton (1974) and Longuet-Higgins (1975) show that the highest wave is not the wave with most mass, momentum, or energy. This is very relevant to both wave breaking and to actually producing a highest wave. Presumably, similar results are likely to hold for rotational waves.

D. Stability

Except for flows in thin films, which are not considered in this work, the currents of interest are turbulent flows, so discussion of stability may seem inappropriate. This is not so. If an instability transfers energy to surface wave motion, then it is of direct interest in the present context. Also it has been suggested, with respect to other turbulent flows, that "instabilities" of the mean velocity may determine the large-scale structure of the flow (Landahl, 1967).

The stability of inviscid flows $(U(z), 0, 0)$ over a rigid bottom to infinitesimal disturbances is considered by Yih (1972). He extends several theorems for flow between two rigid walls to this case and shows that the requirements for stability are very similar in that unstable modes are associated with inflection points in the velocity profile.

Silcock (1975) examines the surface jet flows

$$U = \operatorname{sech} z \qquad \text{and} \qquad U = \exp(-\tfrac{1}{2}z^2) \tag{4.101}$$

in detail and computes the growth rates of infinitesimal disturbances. The stationary-wave solutions form part of the stability boundary, but for small Froude numbers (based on the jet "thickness") the growth rates of the associated instabilities are very small. There are also instabilities that have little effect on the surface.

Unlike many topics in this paper, there are some experimental results. Sarpkaya (1957) reports on a substantial experimental work in which waves were propagated against a stream flowing under gravity. Measurements were made of wave phase velocity, amplitude, wavelength, and shape, for those particular waves that propagated unchanged in amplitude. Higher and shorter waves were amplified, and smaller and longer waves were damped. Figures 19 and 20 are taken from Sarpkaya (1957) and summarize some of the results. It may be noted that only waves of finite amplitude are amplified. It is odd that a set of neutral waves, varying in frequency, was not found for each flow.

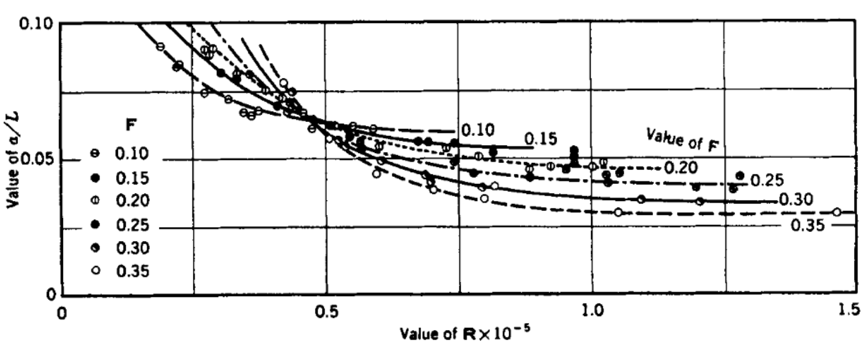


FIG. 19. Stability boundaries of stream flows of different Froude numbers with waves propagating upstream. Amplitude/wavelength is plotted against Reynolds number. (From Sarpkaya, 1957, Fig. 2, p. 575.)

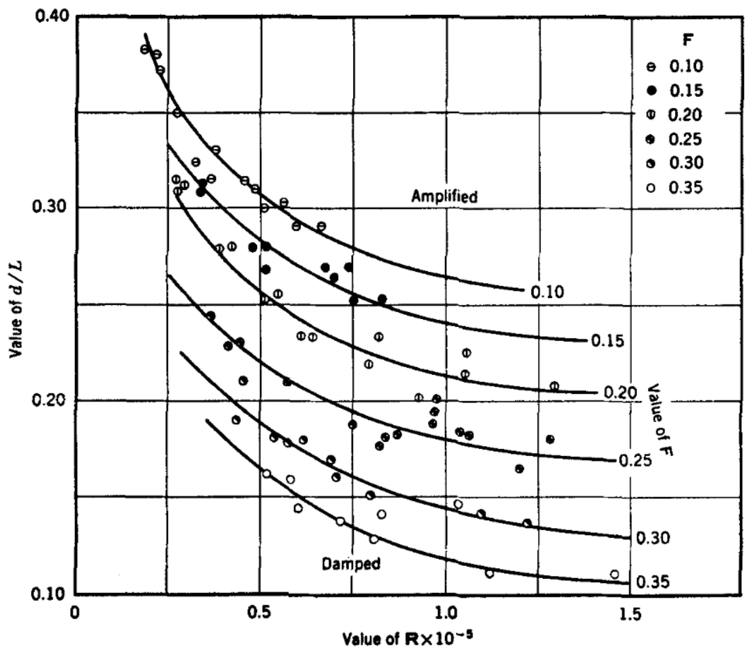


FIG. 20. Stability boundaries of stream flows of different Froude numbers with waves propagating upstream. Depth/wavelength is plotted against Reynolds number. (From Sarpkaya, 1957, Fig. 3, p. 575.)

It is of particular interest that there is no critical layer for any of the waves measured in these experiments; thus an explanation may need to include the interaction of the water waves and the turbulence in the flow, or interaction with the boundary layer on the bed of the channel. The latter seems most likely. By using the results shown and linear irrotational theory it is possible to work out $u(-h)/\bar{U}$ (that is, the particle velocity due to the waves at the bed divided by the mean velocity). For a high proportion of the experimental results, this ratio lies in the range 0.6–0.7 with no systematic variation apparent. This may well be sufficient to cause separation or substantial thickening of the bottom boundary layer.

A more marked instability occurs in uniform streams at high Froude numbers. Large-amplitude waves form and develop bores at their fronts. These progress downstream with variable frequency and amplitude. They are called roll waves. This instability may be demonstrated theoretically (Jeffreys, 1925) by using the linear long-wave equations for irrotational flow and adding a Chézy friction term, that is, a quadratic resistance term that also varies inversely with depth of water. Dressler (1949) gives more details of solutions and Dressler and Pohle (1953) consider more general friction laws. Experimental measurements of the development of roll waves are given by Brock (1969), and numerical examples are calculated and discussed by Jolly and Yevjevich (1974).

Friction laws for streams are one way of representing the turbulence that also gives rise to the mean velocity profile. Laminar flow down a plane is also unstable if the Reynolds number is greater than

$$\tfrac{5}{6}\cot\beta, \tag{4.102}$$

where β is the inclination of the plane to the horizontal, and Benjamin (1957) points out the analogy with roll waves. More details may be found in Yih (1969, Ch. 9, Sect. 9) and experimental results in Benjamin (1961). Corresponding calculations for turbulent stream flow would require an eddy viscosity or other hypothesis to represent the turbulent Reynolds stresses.

Another type of instability gives rise to the surface shear wave in backwash on a beach, mentioned at the end of Section IV,C,3 (Peregrine, 1974). When the backwash is not affected by a following wave it usually forms a nearly stationary turbulent bore where it meets the still water. After a time a long smooth wave may emerge in front of the bore and travel upstream, gaining height, usually until it dwarfs the bore it sprang from. If a hypothesis of a separation of the flow from the bed is correct, the wave may start in the following way. According to linear theory, a small disturbance, exponentially decaying upstream, may precede the bore. If this disturbance is sufficient to cause flow separation on the bed, the wave may appear. It will grow by entrainment of water into the separation "bubble." It seems possible that such an instability may only occur for certain velocity profiles, e.g., flow that starts from rest on a slope, or may depend on the Reynolds number of the flow.

E. Waves on Flow in Channels

This pertains to waves on a flow

$$\mathbf{U} = (U(y, z), 0, 0), \tag{4.103}$$

confined in a channel. Peters (1966) treats the case of long waves, deriving the equation

$$\iint_S dy\, dz/(U(y, z) - c)^2 = b/g \tag{4.104}$$

for the long-wave velocity, in which S is the cross-sectional area of the channel and b its surface breadth.

The linearized equation corresponding to Eq. (4.7) is

$$\frac{\partial}{\partial y}\left[\frac{1}{(U-c)^2}\frac{\partial p}{\partial y}\right]+\frac{\partial}{\partial z}\left[\frac{1}{(U-c)^2}\frac{\partial p}{\partial z}\right]-\frac{k^2}{(U-c)^2}p=0, \qquad (4.105)$$

with

$$\partial p/\partial n = 0 \qquad (4.106)$$

on the walls of the channel and

$$g\, \partial p/\partial z = k^2(U-c)^2 p \qquad (4.107)$$

on the mean free surface. No solutions have been found with any y variation.

Peters (1966) also finds the equation for a solitary-wave solution, but a subsidiary partial differential equation, similar to (4.107), must be solved in S.

However long the waves are, such solutions are only likely to be of value for channels that do not have a large aspect ratio (width/depth). In the irrotational case, Peregrine (1968) shows that a second long-wave approximation (which is needed for the solitary-wave solution) has a term that increases with the square of the aspect ratio for nonrectangular channels.

V. Turbulence

In considering the interaction of waves and turbulence, a useful way to develop ideas is to take a simple view of the turbulence. That is, characterize the turbulence by a length scale and a typical maximum fluctuation of velocity. Although turbulence has important small-scale properties, it seems likely that interactions are dominated by the most prominent turbulent motions, and the ratio of their lengths and velocities to those of the waves.

Perhaps the simplest case to understand, though a difficult one to analyze, is when the turbulence has a scale much greater than the waves and velocities comparable with the wave group velocity. The waves are then refracted in accord with the equations derived in Section II,C. However, consideration of the solutions in Sections II,D and E shows that unless the waves are reflected, by refraction, out of the region of turbulence, they are likely eventually to encounter currents causing their wavelength to diminish considerably so that a high proportion of their energy is lost by breaking. Thus large-scale turbulence acts as a wave absorber. Such behavior is easy to see on rivers, where wind waves get little chance to grow if the turbulence is strong enough. Usually this seems to coincide with a level of turbulence, which noticeably deforms the free surface so that its dominant features are small ripples and dips above vortex cores.

Much stronger turbulence leads to relatively violent surface motions that can generate some propagating waves, or in extremes as in turbulent hydraulic jumps, it leads to the surface breaking up into drops and irregular masses of water. The whole range of behavior may be observed, on a relatively small scale, in the boundary layer of ships.

An example that is commonly observed is the effect of a ship's wake on short wind waves that are incident upon it. The turbulence is relatively strong and often of larger scale and thus acts to absorb or reflect the waves by refracting and steepening them. However, the mean motions associated with a wake are also likely to be important. There is the flow along the wake in the direction of motion of the generating vessel and also the transverse motions due to trailing vortices (from bilges or propellors or both). That these latter may be dominant is indicated by the relatively stronger effect that a curved wake has on waves.

When the turbulent velocity fluctuations of large-scale turbulence are weak compared with the wave velocity, one may think in terms of waves being scattered by the turbulence. Phillips (1959) attempts to analyze this scattering for very weak turbulence, using a Fourier decomposition of the velocity field. It is difficult to make use of such an approach since the components of a spatial Fourier decomposition of turbulence are virtually unknown except for very special cases. Phillips uses estimates based on the inertial subrange of turbulence, but the author does not think this is likely to be an important part of the interaction. It seems somewhat more likely that Howe's (1973) method for treating scattering may be applicable.

Turning to the other extreme, we have small-scale turbulence. This may actually be the same turbulent flow but viewed with respect to different, longer waves. Small-scale turbulence is more amenable to study in laboratory experiments and to the extension of empirical methods established in other fields. For example, the "friction laws" established for steady flows in channels are often extended to unsteady flows such as long waves (see the case of roll waves mentioned in Section IV,D). On the other hand, Sarpkaya's (1957) experiments (also mentioned in Section IV,D) show that some waves on a turbulent flow are amplified.

Taking another viewpoint, the rate of strain tensor for a plane irrotational wave train is

$$e_{ij} = \partial^2\phi/\partial x_i\, \partial x_j\,. \tag{5.1}$$

This is oscillatory, but as Phillips (1959) points out, the Stokes' drift may be more important. More precisely, the finite-strain tensor has a component of second order in ak increasing linearly with time. It acts to stretch vortex lines and thus may lead to stronger interactions than the oscillatory part. Experiments by Green *et al.* (1972), described below, appear to give some support to this idea. If it is important, then the common eddy viscosity hypothesis may be of limited value.

A few experiments have been performed to measure the scattering and dissipation of water waves by turbulence. In none of these experiments has scattering attributable to the turbulence been detected. The most interesting, from the viewpoint of the *rest* of this paper, are two experiments Savitsky (1970) reports. In both experiments turbulence was generated by towing a grid in a ship model testing tank. Waves were sent along the tank to overtake the grid and its turbulence. In the first experiment the grid spanned a 12-ft (3.7 m) tank. In the second experiment, the grid was only 3 ft (0.9 m) wide in a 75-ft (23 m) tank. Savitsky was unable to detect any scattering or dissipation by the turbulence, since in both cases the effects of the mean currents set up by the moving grid dominated the wave behavior. In the first experiment, there was a velocity defect at each side of the tank and the waves became unsteady with curving crests. In the second experiment the wake-type flow refracted and diffracted the waves. The maximum mean velocities in these experiments were more than 10% of the group velocity of the waves.

A more successful experiment is reported by Green *et al.* (1972). In a laboratory tank, turbulence was generated by a grid oscillating vertically. A false bottom was inserted, for all but the longest waves, to protect the surface layer of water from mean currents. The turbulent eddies had a scale of around 1 cm and the shortest waves had a wavelength of about 5 cm. Damping of the waves was observed. Measurements were made on various waves before and after propagating through the turbulent region, the turbulence being approximately the same in all cases.

The results are presented in two different ways, one assuming an exponential decay with distance, the other assuming a quadratic decay law

$$da/dx = -\gamma a^2 \tag{5.2}$$

for the turbulent damping. The coefficients obtained from the measurements show appreciable scatter, but assumption (5.2) gives the smaller scatter. The coefficient γ is found to depend on frequency in such a way that

$$\gamma = C\sigma^5, \tag{5.3}$$

where C is a dimensional constant. Green *et al.* (1972) note Phillips' (1959) suggestion that the most effective interaction with the turbulence may be of second order in ak. Also, one may note that the time that any wave group spends in the turbulent region is inversely proportional to its group velocity. For this purpose it is adequate to assume the linear deep-water gravity wave dispersion relation, in which case Eqs. (5.2) and (5.3) may be rewritten

$$da/dx = -Cg^3a^2k^2/2c_g\,, \tag{5.4}$$

which supports the hypothesis that second-order effects are relevant.

A rather different experiment is reported by Green and Kang (1976). In this case, a single long wave, the fundamental resonant mode of a wave tank, was allowed to decay in the presence of different intensities of turbulence. The turbulence was due to thermal convection generated by heating the bottom of the tank. The intensity of this turbulence depends on the Rayleigh number. A major problem in interpreting the results of this experiment is that less than 20% of the observed damping could be ascribed to turbulence. However, after careful analysis and estimation of the other dissipative effects, Green and Kang present results for the turbulent damping that show an appreciable dependence on Rayleigh number. They also provide an interpretation of the interaction.

Convective turbulence takes the form of intermittent thermals arising from the bottom boundary layer. When there is a horizontal flow, such as that due to long waves, a rising thermal carries relatively stationary fluid from the bottom boundary layer into the main moving mass of fluid. This can be interpreted as giving a Reynolds stress approximately equal to $n\rho uw$, where n is the fraction of the horizontal area over which thermals occur at any instant, u the horizontal velocity of the main mass of fluid, and w a typical vertical velocity in a thermal. Green and Kang's (1976) results are consistent with the estimate of this Reynolds stress that they make.

VI. Ship Waves

A major aim in the study of ship hydrodynamics is the prediction of the total hydrodynamic resistance of a ship. This is a difficult and complex problem, so that another more practical topic is also studied: how to relate measurements on ship models to the behavior of the prototype. The traditional approach has been to divide the resistance into two parts, "viscous resistance" and "wave resistance," and to scale the first according to the Reynolds numbers of the ship and its model and scale the second with their Froude numbers. A similar approach is used for studying the primary problem of predicting ship resistance theoretically.

Independent methods of measuring the viscous and wave components of resistance, by measuring the water velocities and wave amplitudes behind ship models, have shown that this simple view is inadequate. Explanation of the actual resistance involves consideration of the interaction between waves generated by a ship and the flow around and in the wake of the ship. Important papers illustrating this point are those by Lackenby (1965) and Shearer and Cross (1965). The wave resistance of ships is the subject of a recent substantial survey by Wehausen (1973), which gives more details on many of the topics mentioned here.

There are different ways of looking at this subject. One can look at physical quantities and interpret them directly, or, from a theoretical viewpoint, start with some approximation and interpret higher-order terms as interactions. For example, a direct physical approach to a finite-amplitude water wave does not lead to the concepts of linear and nonlinear waves that arises from mathematical approximations. However, interpreting and estimating the physical quantities depends on adequate theoretical backing.

A "physical" analysis of ship resistance is illustrated in Fig. 21, which is an adaptation and extension of a diagram in Brard (1972, Fig. 2). Some of the subdivisions in the diagram are not easy to define, especially with respect to form drag, but it does help to understand the exchanges between the viscous and wave components. Two different ways these may be defined are indicated by the dotted lines. However, in neither case can the viscous component be expected to be entirely independent of the Froude number or the wave component independent of the Reynolds number.

The direct viscous drag is approximately dependent on the wetted surface area, but this depends both on the shape of the water surface around the ship and on the trim of the ship. Both of these depend on the waves generated by the ship in its own vicinity. In the same way the form drag, which may be largely due to regions of flow separation and to trailing vortices, depends on the same two "wave" variables, waterline and trim.

When waves break, as is very often the case near the bows of ships, momentum is transferred from the wave motion into the water. Even for steady ship motion, breaking can be an unsteady process in which case there can be momentum transfer into other wave components. For full-size ships, breaking is probably the most important form of wave dissipation, but there is also some due to turbulence in the boundary layer and wake and due to viscosity. At the lower Reynolds number of ship models, viscosity may be more important and surface tension effects near wave crests can also be relevant.

The current field associated with flow around a ship may also be considered a wave generator. It is propagating into still water at the same speed as the ship. This notion is given approximate quantitative form in the papers by Beck (1971), Brard (1972), and Tatinclaux (1970). There is some imprecision here since this effect could also be termed a wave–current interaction. Most wave–current interactions involve a transfer of momentum between the two components. The mean flow and waves are steady in a frame of reference moving with the ship, and as the work in Section II indicates, in a steady situation wave action flux is conserved in those cases where it can be defined; but conservation of wave action does not normally imply conservation of momentum in one part of the system. The waves generated by the ship interact with the flow around it, the approximately inviscid flow, as well as the boundary layer and wake.

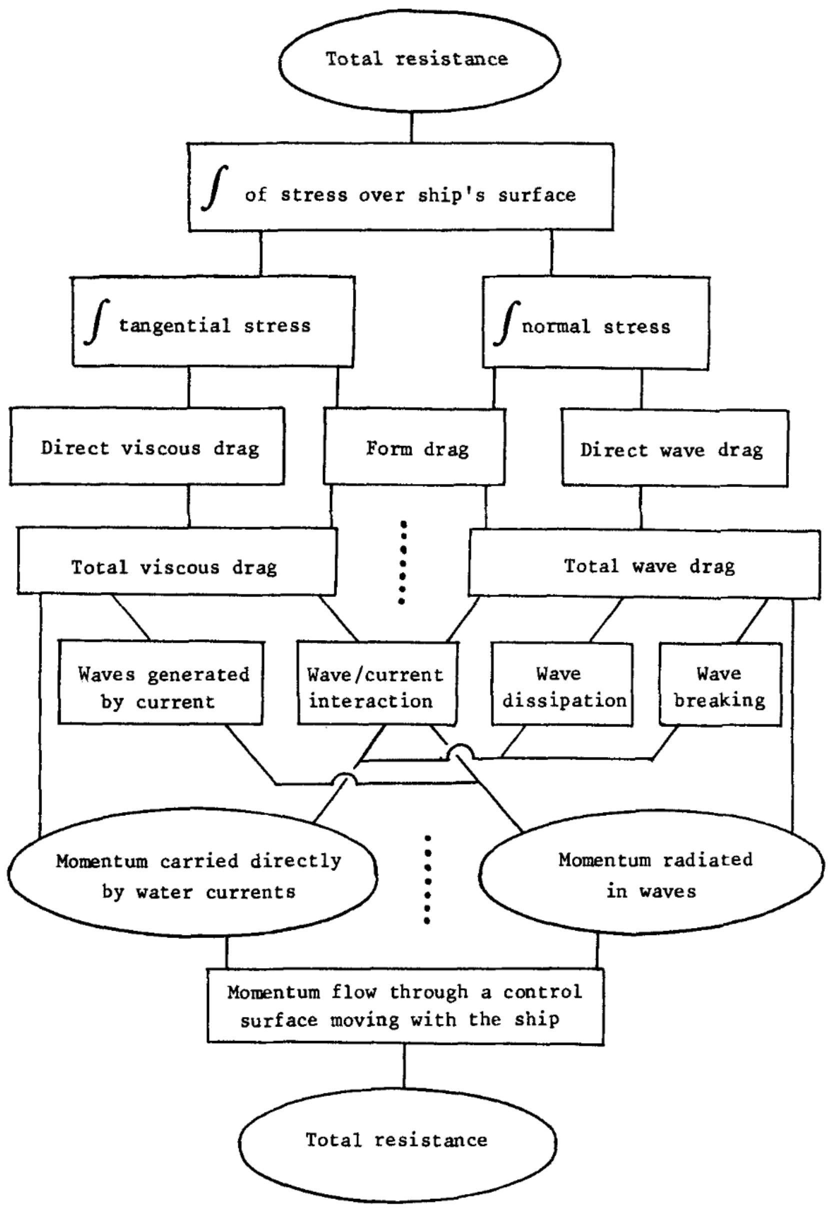


FIG. 21. An analysis of the hydrodynamic resistance to a ship's motion. The quantities within ovals are often measured for ship models. The dotted lines indicate two different divisions into viscous and wave resistance.

While the mathematical problem to be solved in predicting ship resistance can be stated, its solution requires a number of substantial approximations and simplifications. The usual approximations are to assume inviscid irrotational flow, a linearized free-surface boundary condition, and some geometrical ratio, such as breadth/length, to be small. Such approximations may be thought of as a first term in a perturbation expansion, but with so many parameters in the problem it needs careful analysis when looking for higher approximations to ensure a consistent approach. This is discussed in Wehausen (1973), so only the current–wave interactions are mentioned here.

There are two particularly important aspects in the mathematical problem of the interaction of the waves generated by the ship and the flow around it. One is the interaction with the potential flow, and the other is the interaction with the boundary layer and wake.

If the inviscid problem were solved without any approximation involving the speed and shape of the ship, then the first-named interaction would be automatically satisfied. However, this is not usually the case. (Note that unless the ship is very slender or deeply submerged, linearizing the free-surface boundary condition is a poor approximation in the vicinity of the ship.) The ship is usually taken to be "slender" or "thin," or else the Froude number is supposed to be small or large. The thin-ship approximation does take some wave interaction into account at first approximation; the sinkage and trim of the vessel may be calculated. However, the flow around (or perhaps one should say along) the ship and the wave motion are both small, so that interactions come in at the second approximation. On the other hand, if the ship is assumed to have finite bulk and a low Froude number, the first approximation has no waves, so the second approximation necessarily includes solving the wave pattern on the flow field of the first approximation. Dagan (1972) gives an interesting account of this last type of problem, using as a simple example a two-dimensional submerged body. Submerged bodies introduce some further considerations (e.g., see also Farell and Guven, 1973) but two-dimensional potential flow is much simpler than flow in three dimensions.

The interaction of waves with the turbulent flow in the boundary layer and wake is also a difficult problem, even when only mean flows are considered. The approach of representing both the boundary layer and wake by a corresponding displacement thickness, that is, taking a semiinfinite body that is an appropriate amount larger than the ship, has been tried several times with only a relatively small improvement in the results. However, other approximations made at the same time may be more important.

Another approach is to assume that the flow is inviscid but has a vorticity distribution that is chosen to model the actual flow. This seems better than introducing a simple eddy viscosity since the effect of the eddy viscosity on the waves may not be representative of the effect of the effect of the turbulence (see Section V). The simplest example is to consider a wave-making source at the center of a "wake" that is uniform in the direction of motion of the source. Peregrine (1971) uses this model and simplifies the analysis by supposing the waves are so short that the "ray theory" of Section II applies. The resulting wave pattern differs from that for a point source in motion through water at rest. The envelope of wave cusps is inside the Kelvin angle of $19\frac{1}{2}°$ once the maximum wake velocity is greater than 0.1 times the velocity of the wave-making source; the transverse waves behind the source are strongly distorted, although since these are the longest waves the approximation is less likely to be accurate. No information is given about wave amplitudes and with any ray theory approach to the problem initial values on a ray for the wave amplitude are difficult to ascertain. However, the approach is easy to understand.

Rather more detailed descriptions of wake flow are used by Tatinclaux (1970) and Beck (1971). The former has a distribution of vorticity to represent the wake behind a thin vertical two-dimensional cylinder of ogival cross section. Beck uses vortex sheets to model the wake behind a thin ship. Both papers assume that fluid velocities are small and use linearized boundary conditions at the free surface. The contribution of the wake to the wave resistance is calculated for one ship in each paper.

Tatinclaux (1970) chooses a particular vorticity distribution, which decays relatively rapidly behind the cylinder, and calculates solutions for a range of Froude numbers. The wake has most effect for Froude numbers less than 0.5. It increases the wave resistance by an amount that varies considerably with Froude number, in an oscillatory manner, from over $+10$ to -35%. Beck (1971) considers variation of the dimensions of the wake in his model. The effect of his wake is around $\pm 10\%$ of the irrotational wave resistance. All these papers indicate the importance of wave–wake interaction.

A very direct experiment on the interaction of waves and a wake has been performed by Gadd (1975). Two identical ship models were towed in a catamaran arrangement. Where the bow waves of the models intersected, a steep pyramidal wave formed at high enough speeds. A vertical flat plate was introduced along the centerline between the two hulls, so that its trailing edge was just ahead of the steep pyramidal wave. This meant that the bow waves of the twin hulls met the wake of the plate. The introduction of the plate considerably modified the steep wave. It flattened and moved forward the wave peak and caused extensive turbulent flow. A large superficially similar wave often occurs with its crest at a ship's stern. This experiment is expected to give an insight into the interaction between that wave and the ship's boundary layer and wake.

The behavior of this flow brings to mind Banner and Phillips' (1974) paper, which is discussed in Section IV,C. The effect of the reduced velocity in the boundary layer relative to the waves is a decrease in the maximum height attainable. For example, if the velocity of the flow is one-half the ship's speed, use of Eq. (4.100) shows that the maximum elevation of the water is one-quarter its maximum for irrotational flow. Gadd's (1975, Fig. 3) photograph shows that the wave is breaking. One consequence of this observation is that one must expect nonlinear effects to become important at much lower amplitudes in this type of problem than in cases where irrotational flow is a good approximation.

Further details, including wake traverses and wave measurements, are included in Gadd's (1975) paper. They show more details of the interaction that occurs in this experiment. Interesting points are the appreciable changes in the wake behind the hulls when the plate is introduced and the associated differences in the waterline near the stern of the models and in the waves radiated. These appear to be largely due to the bow wave of one hull influencing the stern of the other and its modification when the plate is introduced.

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