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Spectral and statistical properties of the equilibrium range in wind-generated gravity waves

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Section 1 and 2
presented by Momme Hell

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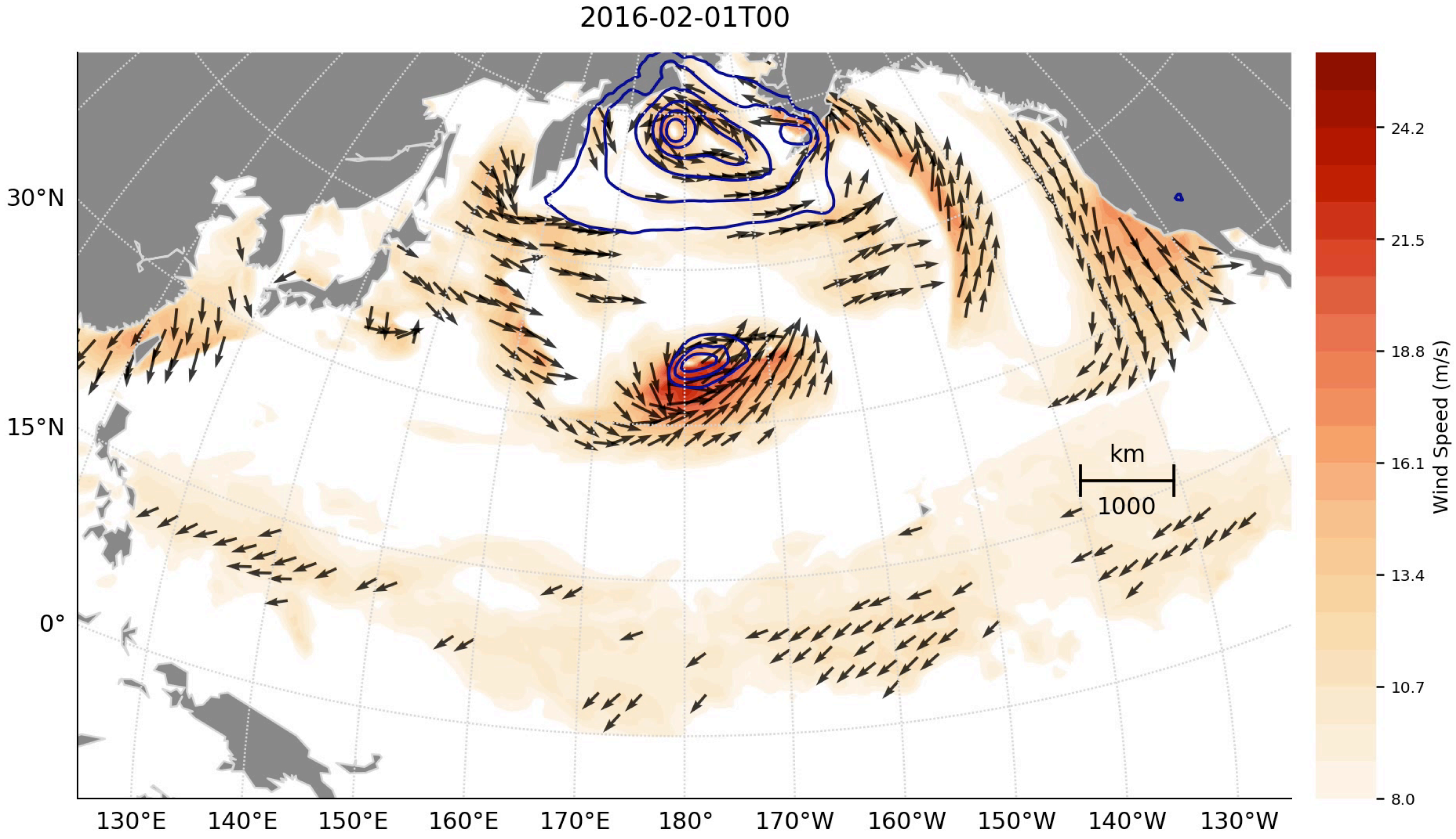
My perspective

Swell waves as remote observation of wave generation and storms



Phillips, 1958 - The equilibrium range in the spectrum of wind-generated waves

An idea of a hard saturation limit rather then one of equilibrium

$$\Psi(\mathbf{k}) \propto f(\theta)\, k^{-4},$$

or

$$\Phi(\sigma) \propto g^2 \sigma^{-5},$$

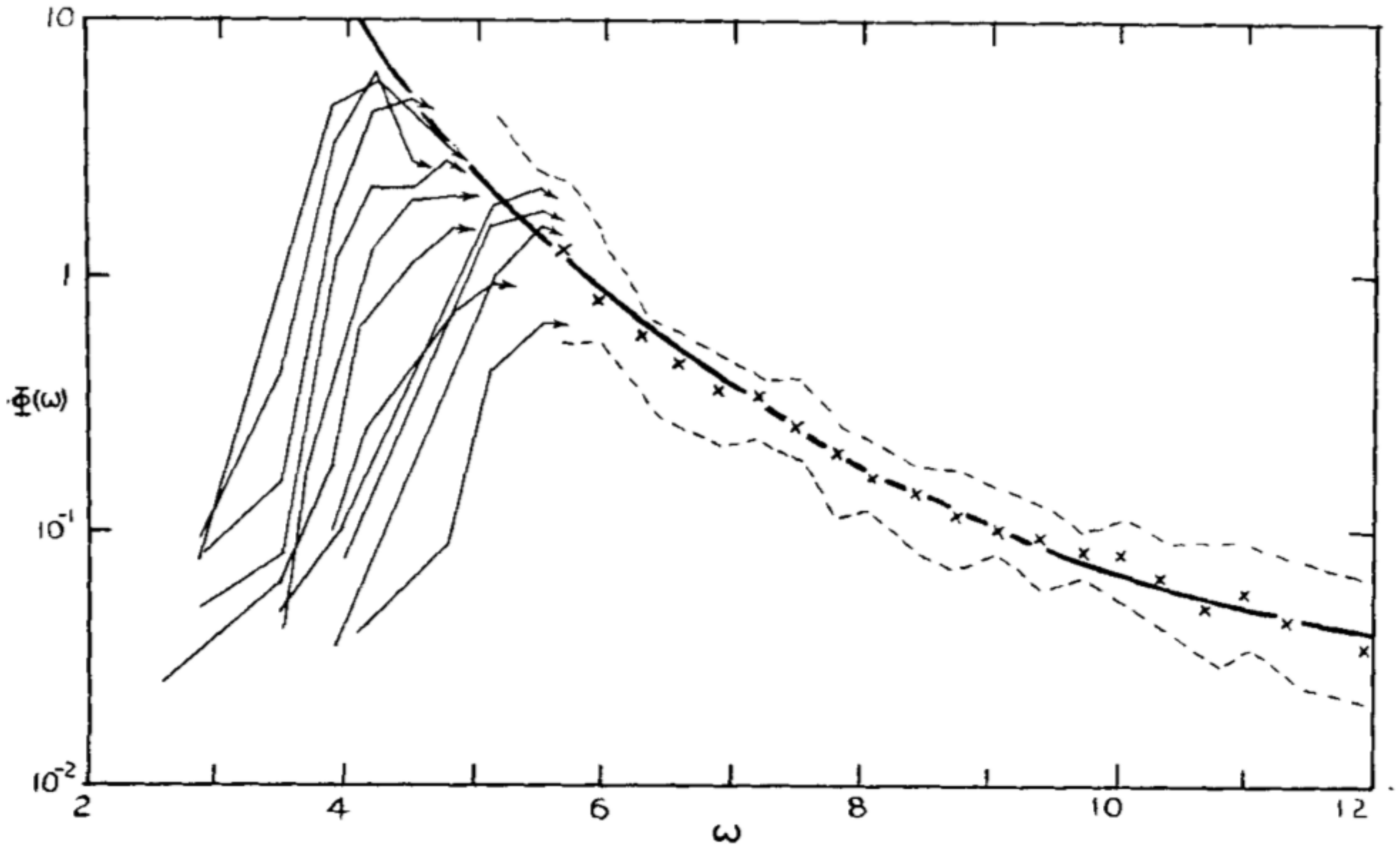


Figure 1. Spectra of wind-generated waves measured by Burling (1955). The cluster of lines on the left are representative of the spectra at low frequencies for which equilibrium has not been attained. On the right the curves merge over the equilibrium range, and the broken lines indicate the extreme measured values of $\Phi(\omega)$ at each frequency ω. The crosses represent the mean observed value at each ω, and the heavy line the relation $\Phi(\omega) = \alpha g^2 \omega^{-5}$ with $\alpha = 7{\cdot}4 \times 10^{-3}$.

- Any excess of spectral energy at a given wave number will be immediately relieved by wave breaking.

Two reason why this is no longer tenable (in 1985).

- There is a fetch dependence (JONSWAP, Hasselmann et. al 1974) if this limit. *How can larger scales (the spectral peak, swell and atmospheric winds) influence the higher wave number spectrum?*
- Wave-current interaction can maintain slopes that are steeper then this limit (interaction with tidal currents)

Alternative representation with dependence on the friction velocity (Toba ,1973)

$$\Phi(\sigma) \propto u_* g \sigma^{-4},$$

Phillips, 1985 is a reformulation of the equilibrium range that takes into account the continuous transfer of energy from the wind, to waves and to ocean.

Phillips, 1985 - Spectral and statistical properties of the equilibrium range

Wave action equation (Phillips, 1977)

Action spectral density N(k)

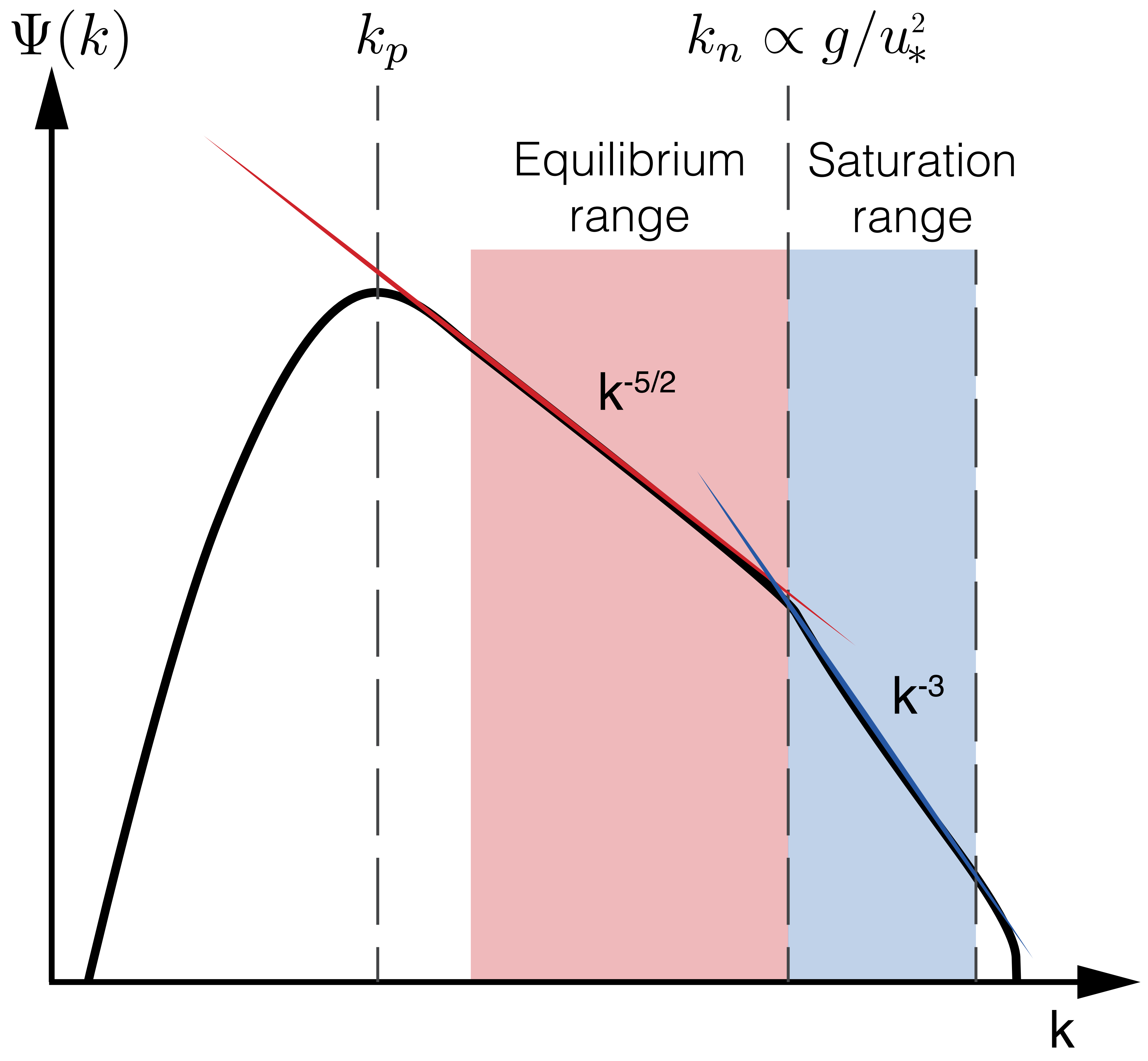
$$N(\boldsymbol{k}) = \frac{g}{\sigma}\Psi(\boldsymbol{k}) = \left(\frac{g}{k}\right)^{\frac{1}{2}}\Psi(\boldsymbol{k}),$$

Is conserved following an energy path

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\partial N}{\partial t} + (\boldsymbol{c}_g + \boldsymbol{U})\cdot\nabla N = \boxed{-\nabla_{\boldsymbol{k}}\cdot \boldsymbol{T}(\boldsymbol{k}) + S_{\mathrm{w}} - D.} = 0$$

In the equilibrium range and under steady, continuous winds, the timescale of wave growth is slow and advection is neglect-able

The source and sink terms are in **statistical equilibrium**



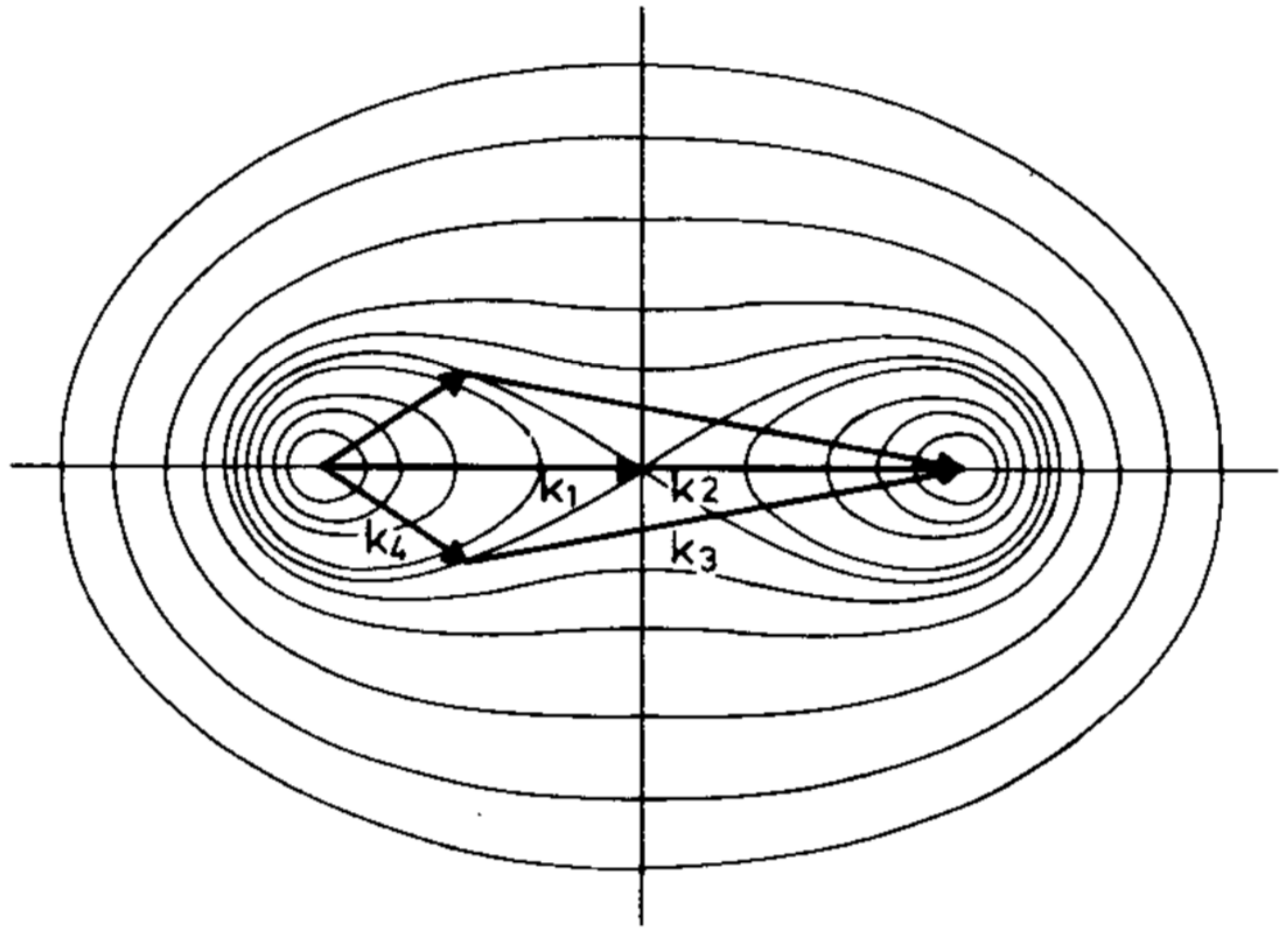
Phillips, 1985 - Spectral and statistical properties of the equilibrium range

Defining the equilibrium

$$-\nabla_k \cdot T(k) + S_w - D = 0.$$

Spectral Flux divergence
by non-linear wave-wave interaction

Solving this is computational expensive. Wave models use simplified parameterizations
Hasselmann, S and Hasselmann, K (1985),
Hasselmann, S et. al (1985)



$$-\nabla_k \cdot T(k) = \iiint Q^2\{[N(k)+N(k_1)]\,N(k_2)\,N(k_3) - [N(k_2)+N(k_3)]\,N(k)\,N(k_1)\} \times \delta(k+k_1-k_2-k_3)\,\delta(\sigma+\sigma_1-\sigma_2-\sigma_3)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3, \quad (2.7)$$

4-wave resonance condition
weakly non-linear and rather inefficient

$$k + k_1 = k_2 + k_3, \quad \sigma + \sigma_1 = \sigma_2 + \sigma_3,$$

redefine 2.7 in terms of the (dimension less) **degree of saturation B(k)**

$$-\nabla_k \cdot T(k) \propto g k^{-4} B^3(k).$$

Dominant interaction are primary local, that is, its Q is larges when all k have similar length (with a factor of 2),
(Sell & Hasselmann, 1972; Fox, 1976; Longuet-Higgins, 1976).

At the spectral peak, energy goes mainly to wave numbers in tis vicinity.

In the equilibrium range, the spectral energy flux is determined by the local spectral density N(k), such that

$$-\nabla_k \cdot T(k) \propto Q^2\, N^3(k)\, k^4/\sigma, \quad \propto g^{-\frac{1}{2}} k^{\frac{19}{2}} N^3(k),$$

with looking at 2.7 ~ N^3 and Q ~ k^6

$$B(k) = k^4\, \Psi(k) = g^{-\frac{1}{2}} k^{\frac{9}{2}} N(k)$$

Phillips, 1985 - Spectral and statistical properties of the equilibrium range

Defining the equilibrium

$$-\nabla_k \cdot T(k) + S_w - D = 0.$$

Rate of action input by wind

$$\tau = \frac{\rho_a}{\rho_w} u_\star^2 \; ak \cos(\phi)$$

- *a k* to account for ocean surface tilt
- *cos(phi)* to account for the angle between wind and waves

Rate of work of the ocean surface $S_w = \vec{\tau} \cdot \vec{U_o}$

with $U_o = (a\, k)\, c$, we get

$$S_w \propto \frac{\rho_a}{\rho_w} u_\star^2 \; (ak)^2 \cos(\phi) \; c$$

$$S_w \propto \frac{\rho_a}{\rho_w} \cos\theta \, u_*^2 k^2 c \sigma^{-1} \Psi(k),$$
$$= \frac{\rho_a}{\rho_w} \cos\theta \left(\frac{u_*}{c}\right)^2 \sigma N(k),$$

$$S_w = m \cos^{2p}\theta \, g k^{-4} \left(\frac{u_*}{c}\right)^2 B(k).$$

- added a stronger angle dependence
- added an empirical constant m
- expressed N(k) again as B(k)

Wave growth rate is based on

- the friction velocity u*
- the **degree of saturation B(k)**
- a constant m ≈ 0.04, (0.05, or 0.07 …)
- Valid for c(k) > 10 u*, because atm. critical layer are clearly above the water surface (Miles, 1957).
- For c < 10 u*, critical layers is submerged in the wave field an distorted by capillary waves.

Side note: Parametrization of surface stress

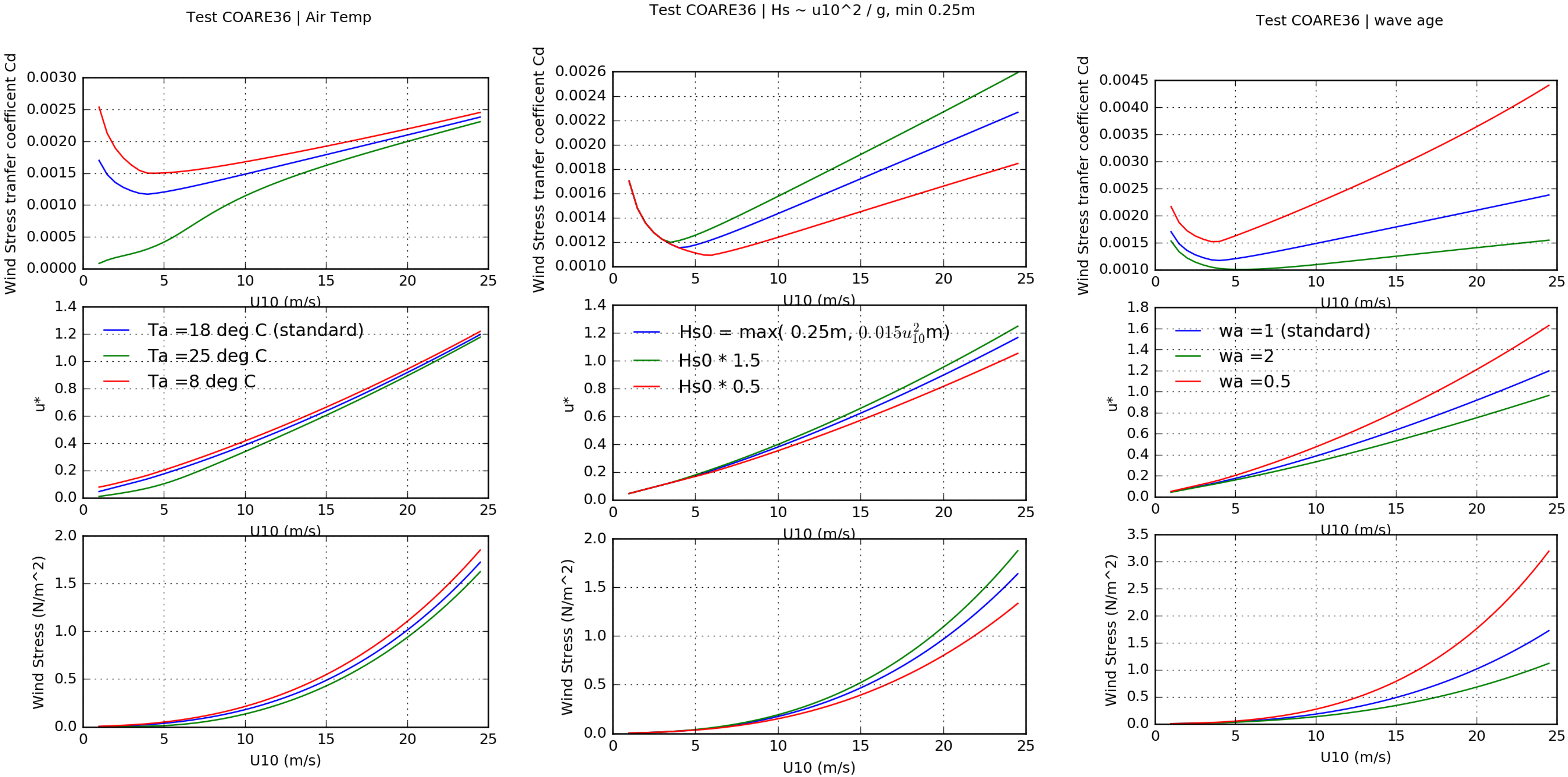
The modern boundary layer parametrization

$$\vec{\tau} = \frac{\rho_a}{\rho_w} \ C_d \ |\vec{u}_{10}|\vec{u}_{10}$$

Satellite scatterometers measure surface roughness and invert for a U10 velocity, assuming a mean square slope (Cox and Munk, 1954) or surface slope spectrum $\Psi(k)$ (Wentz et. al, 1984) and a drag coefficient C_d.

C_d assumes a standard static stability of the boundary layer, a wave age and Hs.

Side note: Parametrization of surface stress (Fairall et al, 2003)



Phillips, 1985 - Spectral and statistical properties of the equilibrium range

Defining the equilibrium

$$-\nabla_k \cdot T(k) + S_w - D = 0.$$

Dissipation by wave breaking

"wave breaking is a local process in physical space and a distributed process in Fourier space" (p. 512)

- The degree of saturation B(k) maybe locally enhanced by the presence of currents or enhanced wind input, but the extent to which wave breaking occurs is (always) a function of the local degree of saturation

$$D = gk^{-4} f(B(\boldsymbol{k}))$$

- This broad spectral process only applies in the equilibrium range (because of assumptions of the shape of B(k)).

What is the functional relation f(B(k))?

It is *simply* what it has to be. The rate of small scale dissipation will adjust to amount energy of what is given down to them by strong non-linear interaction on larger scale (analogy to 3D turbulence in high Reynolds number flows).

Phillips, 1985 - Spectral and statistical properties of the equilibrium range

Constrains and consequences

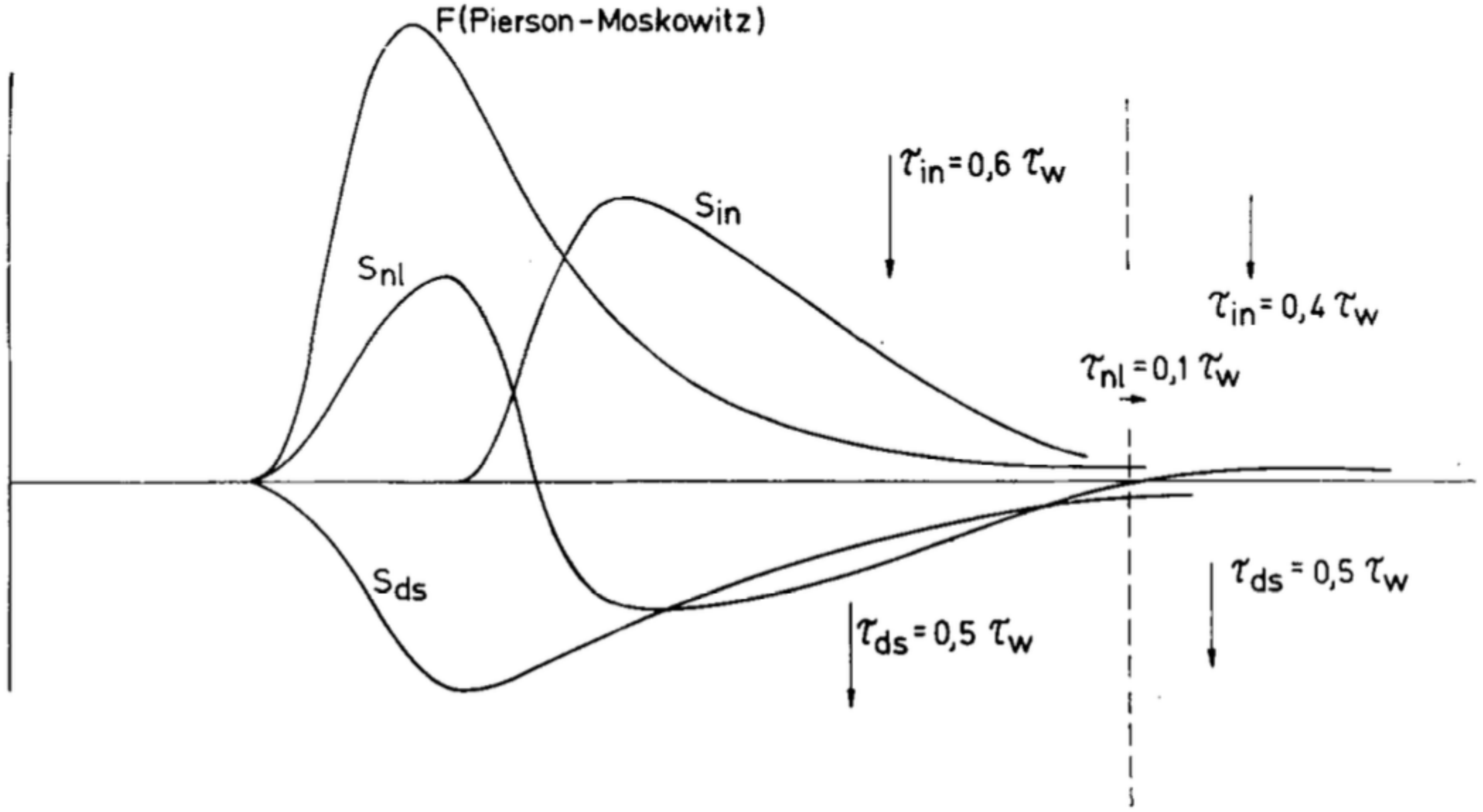


Fig. 3. Energy balance (schematic) for a fully developed Pierson-Moskowitz (1964) spectrum. Frequency and ordinate scales arbitrary.

(Hasselmann, 1974)

- The dominant wind input is at $k > k_p$ and depends on u* (Toba, 1973)
- There is energy spectral flux divergence through out the equilibrium range, independent of the particular shape of the spectrum. (Hasselmann et. al 1973, Komen et. al 1984).
- Dissipative processes are pertinent in the equilibrium range.

All three terms are important, there is no internal wavenumber

—> **the ratio of terms must constant in the equilibrium range**

$$B^3(\boldsymbol{k}) \propto m \cos^{2p}\theta \left(\frac{u_*}{c}\right)^2 B(\boldsymbol{k}) \propto f(B(\boldsymbol{k})).$$

It follows that the degree of saturation and the dissipation can be expressed as a function of u*.

Phillips,1985 - Spectral and statistical properties of the equilibrium range in wind-generated gravity waves

Summary

- There is an equilibrium range where wind input, non-linear wave-wave interaction and dissipation balance and are proportional.
- The majority of the wind input happens in the equilibrium range and *shapes the spectrum.*
- Each term in the equilibrium range depends on the degree of saturation, which is a driven by u*
- the upper limit of the equilibrium range scales with g / u*²