Stokes Drift for Random Gravity Waves

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The Stokes drift velocity for random surface gravity waves is given in terms of the directional energy spectrum. The drift velocity is evaluated for empirical forms of the spectrum for fully developed seas. The result is that the surface drift velocity increases linearly with increasing wind speed and the ratio of surface drift speed to wind speed at 19.5 meters is between 1.6 and 3.6%. The stokes drift velocity may therefore contribute significantly to the total mean surface currents in the ocean.

Introduction. The modified form of the Stokes drift velocity [*Stokes*, 1847] for a random sea is briefly considered. For a statistically stationary and horizontally homogeneous wave field, the total wave drift is the sum of the drifts of the individual wave components. The expression for the Stokes drift, given here in terms of the full two-dimensional energy spectrum for arbitrary constant mean depth, is more general than expressions given by *Chang* [1969] and *Bye* [1967].

Bye [1967] computed the Stokes drift for the equilibrium range (ω^{-5} law portion) of the wind wave spectrum, and *Chin and Pierson* [1969] have computed examples of the surface drift for the whole North Atlantic. The application made here to fully developed seas uses the empirical spectral forms of *Pierson and Moskowitz* [1964], giving the result that the ratio of Stokes drift speed at the surface to wind speed at 19.5 meters is between 1.6 and 3.6%. The Stokes drift decreases rapidly within a few meters of the surface.

Stokes drift. Assume a statistically stationary and horizontally homogeneous wave field and let the surface displacement $\zeta(\mathbf{x}, t)$ be represented by

$$\zeta(\mathbf{x}, t) = \sum_{\mathbf{k}} [\eta_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \eta_{\mathbf{k}}^* e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}] \tag{1}$$

where $\mathbf{x}$ and $\mathbf{k}$ are the horizontal coordinate and wave number vectors, respectively, and $\eta_{\mathbf{k}}$ is a random Fourier amplitude ($\eta_{\mathbf{k}}^*$ is its complex conjugate). The relation of frequency (ω) to wave number (k) is

$$\omega^2 = gk \tanh kh \tag{2}$$

where g is the acceleration of gravity and h is the constant mean depth. The two-dimensional energy spectrum $F(\mathbf{k})$ is defined by

$$\rho g\langle\zeta^2(\mathbf{x}, t)\rangle = 2\rho g \sum_{\mathbf{k}} \langle\eta_{\mathbf{k}}\eta_{\mathbf{k}}^*\rangle = \iint_{-\infty}^{\infty} F(\mathbf{k})\, d\mathbf{k} \tag{3}$$

where ρ is the constant water density and the angle brackets denote the ensemble mean. Owing to the statistical assumptions, the following property holds

$$\langle\eta_{\mathbf{k}_1}\eta_{\mathbf{k}_2}\rangle = \langle\eta_{\mathbf{k}_1}\eta_{\mathbf{k}_2}^*\rangle = 0 \quad \text{for} \quad \mathbf{k}_1 \neq \mathbf{k}_2 \tag{4}$$

The (ensemble) mean second-order drift velocity for the wave field (1) is easily derived by applying the method of *Stokes* [1847] (see also *Phillips* [1966, article 3.3]). This method involves the expansion of the motion of a fluid particle about its mean position. The total mean drift is the superposition of the drifts for each wave component; cross products containing pairs of wave components with different wave numbers do not contribute to the mean drift because of property 4. The resulting Stokes drift velocity $\mathbf{U}(z)$ for the two-dimensional spectrum and for arbitrary depth is

$$\mathbf{U}(z) = 1/\rho \iint_{-\infty}^{\infty} F(\mathbf{k}) \frac{\mathbf{k}}{\omega(k)} \cdot \left[\frac{2k \cosh 2k(z+h)}{\sinh 2kh}\right] d\mathbf{k} \tag{5}$$

where z is the mean vertical position of a fluid particle measured positive upward from the mean surface ($z = 0$). In this form equation 5 is also general with respect to surface tension T, when the right side of equation 2 and the energy spectrum defined by (3) are multiplied by the factor $1 + Tk^2/\rho g$. The conditions under which (5) is valid are

$$k\,|\eta_{\mathbf{k}}| \ll 1 \qquad |\eta_{\mathbf{k}}|/h \ll 1 \tag{6}$$

i.e. the wave amplitude must be small compared with both the wavelength and the water depth for each wave component. For a line spectrum equation 5 reduces to the expression given by *Stokes* [1847] for a single wave component.

The mean momentum per unit area **M** obtained from (5) gives the well-known expression (see *Hasselmann* [1963], for example)

$$\mathbf{M} \equiv \int_{-h}^{0} \rho \mathbf{U}(z)\, dz = \iint_{-\infty}^{\infty} F(\mathbf{k}) \frac{\mathbf{k}}{\omega}\, d\mathbf{k} \tag{7}$$

It can be seen from (7) that for a given energy spectrum the mean momentum per unit area, and therefore the Stokes drift at each level, increases as the depth decreases, because for a given wave number the wave phase velocity decreases with decreasing depth (see equation 2).

Application. An estimate of the magnitude of the Stokes drift for the deep ocean can be obtained from (5) by using empirical energy spectra for fully developed seas. For this purpose it is convenient to write (5) in terms of the one-dimensional spectrum $f(\omega)$ in the deep water limit ($kh \rightarrow \infty$)

$$U(z) = \frac{2}{\rho g^2} \int_0^{\infty} f(\omega)\omega^3 e^{+2\omega^2 z/g}\, d\omega \qquad \text{for} \quad -\infty \le z \le 0 \tag{8}$$

which follows from (5) by the transformation

$$kF(\mathbf{k}) = \left|\frac{d\omega}{dk}\right| f(\omega) S(\alpha), \quad \int_{-\pi}^{\pi} S(\alpha)\, d\alpha = 1 \tag{9}$$

where it is assumed that all the waves travel in the same direction $\alpha = \alpha_0$, say, and $U(z)$ is the drift in the direction α_0. Equation 8 is essentially the form for the Stokes drift given by *Chang* [1969] and *Bye* [1967].

Pierson and Moskowitz [1964] considered empirical spectra of the form (10) to represent the ocean wave field under fully developed conditions in deep water. They fitted three spectra corresponding to $n = 2, 3, 4$ to wave data for wind speeds W between 10 and 20 m/sec measured at an elevation of 19.5 meters; the form for $n = 4$ gave a slightly better fit than the other two.

$$f_n(\omega) = (\alpha_n \rho g^3/\omega^5) e^{-\beta_n (g/W\omega)^n} \tag{10}$$

The constants α_n, β_n given by (11) were determined in such a way that for a given wind speed the three spectra have the same maximum value at the same frequency. The two remaining constants f_0, ν_0

$$\alpha_n = (f_0/2\pi)(2\pi\nu_0)^5 e^{5/n} \qquad \beta_n = (5/n)(2\pi\nu_0)^n \tag{11}$$

in (11) were given the values $f_0 = 2.75 \times 10^{-2}$ and $\nu_0 = 0.140$, which best fit the data.

The Stokes drift at the surface U_n (0) obtained by putting (10) and (11) into (8) gives the following result:

$$\frac{U_n(0)}{W} = \begin{cases} 3.58\% & n = 2 \\ 2.09\% & n = 3 \\ 1.57\% & n = 4 \end{cases} \tag{12}$$

The values of $U_n(0)/W$ increase as n decreases; this is due primarily to the fact that the total energy of $f_n(\omega)$ increases (the ratio of the total energy of f_2 to f_4 is 1.40 independent of wind speed). *Chin and Pierson* [1969] obtained a similar result for the case $n = 4$, although they plotted the average Stokes drift (over the depth of a few meters) against wind speed.

The Stokes drift as a function of depth $U_2(z)$ can be found in closed form for the spectrum $f_2(\omega)$

$$U_2(z) = A W e^{-B(g|z|)^{1/2}/W} \qquad \text{for} \quad -\infty \le z \le 0 \tag{13}$$

where

$$A = [(2\pi)^7 e^5/5]^{1/2} f_0 \nu_0^4 = 3.58 \times 10^{-2}$$

$$B = 4\pi 5^{1/2} \nu_0 = 3.93$$

The Stokes drift $U_2(z)$ in (13) is illustrated in Figure 2 for the three wind speeds $W = 10$, 15, 20 m/sec, and the corresponding spectrum

$E_2(\nu) = (2\pi/\rho g)\, f_2(\omega)$ $(\nu = \omega/2\pi)$ is shown in Figure 1 for the same three wind speeds. The drift velocities $U_3(z)$ and $U_4(z)$ (not shown) corresponding to $f_3(\omega)$ and $f_4(\omega)$ were computed numerically and are qualitatively similar to Figure 2; i.e., the surface values are lower as given by (12) but have a similar rapid decrease with depth in the upper few meters. As can be seen from (8), this rapid decrease of the drift velocity with depth is due to the fact that the high-frequency components contribute only in a thin layer near the surface.

Chang [1969] obtained agreement between theoretical and measured values of the surface drift for random waves generated in a wave tank, but measured values of the Stokes drift have not yet been made in the ocean. A review of previous investigations by *Tomczak* [1964] indicates, however, that aside from regions of strong permanent currents, several reported values for the ratio of total surface drift to wind speed vary widely from about 1.4 to 4.3%, the largest values corresponding to currents measured right at the surface, and the smallest values corresponding to currents measured a few meters below the surface. This decrease in the ratio with depth agrees qualitatively with

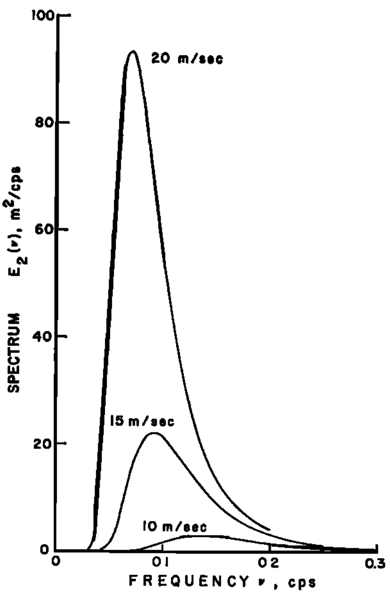

Fig. 1. The empirical spectrum $E_2(\nu)$ as a function of frequency ν for the three wind speeds 10, 15, and 20 m/sec.

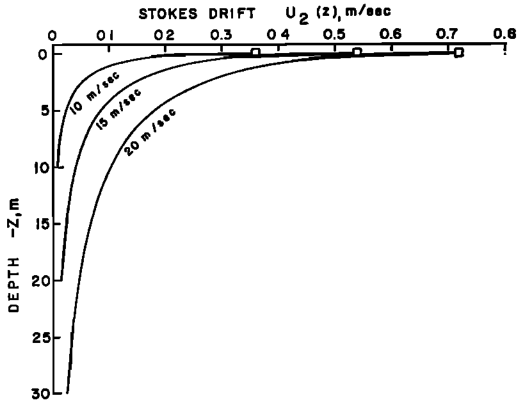

Fig. 2. The Stokes drift velocity $U_2(z)$, based on the spectrum in Figure 1, as a function of the depth z for the three wind speeds 10, 15, and 20 m/sec.

Figure 2 and therefore gives qualitative support for the existence of the Stokes drift in the ocean.

If the surface drift values for a fully developed sea do apply to the ocean, (12) indicates that the Stokes drift could contribute significantly to the total surface drift. However, the values in (12) represent rather extreme estimates for the Stokes drift in the ocean, because fully developed seas occur rather infrequently.

Discussion. The effect of an angular spread in the energy spectrum would be to decrease the Stokes drift component in the principle direction of wave propagation. For example, for the symmetric spreading factor $S(\alpha)$

$$S(\alpha) = \begin{cases} C_m \cos^m \alpha & |\alpha| \leq \pi/2 \\ 0 & |\alpha| > \pi/2 \end{cases}$$

where C_m is determined by (9), the Stokes drift in the direction $\alpha = 0$ is reduced at all depths over values for the unidirectional spectrum $(m \to \infty)$ by the factor 0.91 for $m = 4$ and 0.85 for $m = 2$. Thus, the drift values might be 10 or 15% smaller than the values given by (12) and (13).

The effect of finite depth has not been considered, but, based on (8), (9), and the discussion following (7), it may be anticipated, that in general the Stokes drift values at the surface computed in (12) would be increased if finite depth were taken into account, provided that the empirical spectra (10) with the constants (11) also hold for finite depth.

In applying equation 5 to the ocean, the fact that the integration in (8) was extended to infinite frequencies, whereas the spectral form is not well known for frequencies greater than about 10 Hz (above which surface tension effects become important), probably does not effect the values in (12) and (13) very much. For example, the contribution to $U_2(0)$ from frequencies in the range 10 Hz $\leq \nu \leq \infty$ is only 2.4% for a wind speed of 10 m/sec and 1.2% for a wind speed of 20 m/sec.

Friction may have an important influence on the Stokes drift, and this influence has not been taken into account. However, the results of *Longuet-Higgins* [1953] and *Chang* [1969] indicate that the surface drift may be nearly unchanged by friction, although the gradient of the drift velocity near the surface may be doubled.

A more serious limitation on the application of (5) to the ocean is that for fully developed seas wave breaking occurs, and this violates the first condition in (6) under which (5) was derived. It is also unclear to what extent the irrotational theory, on which (5) is based, holds for an active wind wave field. Therefore, the values in (12) and (13) should be considered only as estimates of the Stokes drift velocity for fully developed conditions in the deep ocean.

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