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Recent laboratory and field observations suggest that the effects of nonlinearity and dispersion may be significant for the evolution of Kelvin waves in rotating channels. Here, it is shown that a pair of Kelvin wave modes may interact with a third Poincaré mode to form a resonant triad. The evolution of the triad is governed by the well-known three-wave interaction equations. The implications of this instability for naturally occurring Kelvin waves are discussed.

I. INTRODUCTION

In recent years considerable interest has developed in the evolution of weakly nonlinear and dispersive internal gravity waves in a rotating frame of reference. For the nonrotating case the theory is well developed, the evolution being governed by the one-dimensional Korteweg–de Vries equation. However, Maxworthy[1] pointed out that for internal wave motions in many of earth's lakes, sea straits, and coastal regions, the transverse scale is not negligible when compared to the Rossby radius, showing that rotational effects cannot be ignored.

Much of the theoretical and experimental work has been on the evolution of solitarylike nonlinear Kelvin waves in a channel. In a separate paper (Melville, Tomasson, and Renouard[2]) these waves are shown to be unstable due to a direct nonlinear resonant interaction with the linear Poincaré modes of the channel, the resonance being possible because of the positive speed correction of the nonlinear Kelvin wave relative to the linear mode. A result of this resonant interaction is an attenuation of the Kelvin wave as it propagates along the channel, together with the evolution of a curved crest of the leading wave. This is qualitatively consistent with experimental results by Maxworthy[1] and Renouard, d'Hiéres, and Zhang.[3]

This work considers a second triad resonance which may occur for weaker nonlinearity. Thus we take the first-order solution to consist of periodic linear dispersive Kelvin waves, rather than a nonlinear solitary wave. Because of the weak dispersion of the linear modes, when expanded in the nonlinear parameter, resonant triads are possible between two Kelvin modes and a Poincaré mode, resulting in the instability of the Kelvin modes for some cases of interest. This resonance differs from the one described above (Melville, Tomasson, and Renouard[2]) in that here the two Kelvin waves involved are periodic, linear modes as opposed to nonlinear wave components of a solitarylike wave. The possibility of the triad resonance is due solely to the weak dispersion of the three linear modes, whereas the direct resonance is primarily a result of the nonlinear speed correction of the Kelvin wave, although slightly modified by the weak dispersion of the linear Poincaré modes.

II. GOVERNING EQUATIONS

The motivation for studying this problem originally came from the dynamics of internal wave motions; however, the phenomena also occur in the simpler corresponding flow of a homogeneous fluid. We therefore consider the motion of a homogeneous inviscid fluid in a channel of width W and constant depth h, rotating with angular velocity $f/2$ about the vertical z axis.

Assuming nonlinear, dispersive, transverse, and rotational effects to be weak, we define three small parameters,

$$\alpha = a/h, \tag{1a}$$

$$\beta = (kh)^2, \tag{1b}$$

$$\Gamma = (l/k)^2 = (f/kc_0)^2 = (kR)^{-2}, \tag{1c}$$

where α is a measure of nonlinearity, β a measure of dispersion, and Γ a measure of transverse variation and rotational effects. Here a is a typical wave amplitude, c_0 is the linear wave speed ($c_0^2 = gh$), k^{-1} is a typical length scale in the longitudinal x-direction, and l^{-1} is a typical length scale in the transverse y direction, equal to the Rossby radius R.

By introducing the scaling

$$\eta = \alpha h \eta', \quad u = \alpha c_0 u', \quad v = \alpha \Gamma^{1/2} c_0 v',$$
$$t = t'/kc_0, \quad x = k^{-1}x', \quad y = l^{-1}y', \quad z = hz', \tag{2}$$

where primes denote the nondimensional variables, and choosing

$$\Gamma = O(\beta) \ll 1, \tag{3a}$$

$$\alpha = O(\beta^{3/2}) \ll 1, \tag{3b}$$

i.e., nonlinearity weaker than dispersive, transverse, and rotational effects, we can derive the following coupled evolution equations:

$$u_t + u_x + \tfrac{3}{2}\alpha u u_x - (\beta/6) u_{xxt} + (\Gamma/2)(v_y - v) = 0, \tag{4a}$$

$$v_t + u + u_y = 0, \tag{4b}$$

with boundary conditions

$$v = 0, \quad \text{at } y = 0, W \tag{5}$$

for waves traveling in the positive x direction. Here (u,v) are the first-order, depth-independent velocities and the primes have been dropped for convenience. The derivation follows exactly the same lines as that described by Grimshaw and Melville[4] and by Melville, Tomasson, and Renouard,[2] except for the slightly different scaling given by Eqs. (3a) and (3b).

With this choice of the scaling, triad resonant interactions between linear, weakly dispersive Kelvin and Poincaré

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modes become possible. In what follows, solutions to these equations are studied, in particular, examining the stability of linear Kelvin modes with respect to the resonant interaction mentioned above. In Sec. III a very simplified problem is considered. By looking at the evolution of the three modes of a triad as an isolated problem, analytical solutions can be developed, which show that for certain cases of interest a pair of linear Kelvin modes may resonantly interact with a third Poincaré mode. In Sec. IV this analysis is extended to include all modes. Approximate analytical solutions are obtained and compared to numerical solutions of the full equations (4a) and (4b).

III. INSTABILITY OF KELVIN WAVES

Eliminating v from the coupled evolution equations (4a) and (4b) gives

$$u_{tt}+u_{xt}+\tfrac{3}{2}\alpha[uu_x]_t-(\beta/6)u_{xxtt}+(\Gamma/2)[u-u_{yy}]=0, \tag{6a}$$

with boundary conditions

$$u+u_y=0,\quad \text{at } y=0,W. \tag{6b}$$

This equation is asymptotically equivalent to the modified Kadomtsev–Petviashvili equation, studied by several other authors (cf. Grimshaw[5]).

Recalling the assumption that nonlinear effects are weaker than dispersive, transverse, and rotational effects [(3a) and (3b)], we expand the solution for u in the nonlinear parameter only, writing

$$u(x,y,t)=u^{(0)}(x,y,t)+\alpha u^{(1)}(x,y,t)+O(\alpha^2). \tag{7}$$

This linear, dispersion solution perturbed in the nonlinear parameter is valid for

$$t\ll O(\beta^2/\alpha^3)=O(\beta^{-5/2})$$

(cf. Mei[6]).

Substituting (7) into Eqs. (6a) and (6b) gives to lowest order

$$\mathscr{L}(u^{(0)})=0, \tag{8a}$$

with the boundary condition

$$u^{(0)}+u_y^{(0)}=0,\quad \text{at } y=0,W. \tag{8b}$$

Here $\mathscr{L}$ is the linear, weakly dispersive, and rotational operator

$$\mathscr{L}=\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x\,\partial t}-\frac{\beta}{6}\frac{\partial^4}{\partial x^2\,\partial t^2}+\frac{\Gamma}{2}\left(1-\frac{\partial^2}{\partial y^2}\right). \tag{9}$$

Solutions of this lowest-order problem satisfy the dispersion relationship

$$\sigma^2[1+(\beta/6)k^2]-\sigma k-(\Gamma/2)(1+l^2)=0, \tag{10}$$

where $u\sim e^{i(kx+ly-\sigma t)}$. The first root of the dispersion relationship has $\sigma=O(1)$ and corresponds to the linear Kelvin ($l=i$) and Poincaré modes ($l=n\pi/W$, $n=1,2,...$) propagating along the channel in the positive x direction. This solution is shown in Fig. 1. The other root has $\sigma=O(\beta)$ and corresponds to the conservation and stationarity of potential vorticity in the exact linear problem. In this formulation these modes propagate slowly to the left unless $(l^2+1)=0$, in which case they are stationary and the primary mode is the linear Kelvin wave.

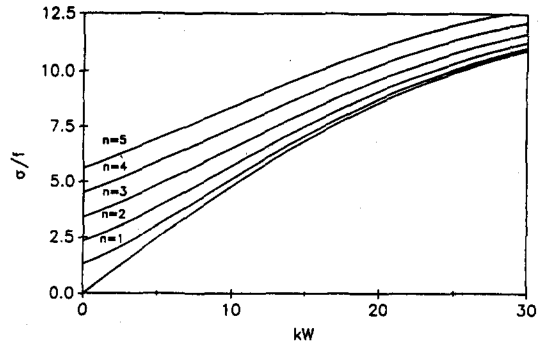

FIG. 1. The dispersion relationship for the linear, nonhydrostatic Kelvin and Poincaré modes ($n=1,2,...$) of the channel.

The weak dispersion of the Kelvin and Poincaré modes gives rise to the possibility of resonant triads, i.e., wavenumber/frequency pairs satisfying

$$\sigma_1+\sigma_2=\sigma_3, \tag{11a}$$

$$k_1+k_2=k_3, \tag{11b}$$

$$l_1+l_2=l_3, \tag{11c}$$

where two Kelvin modes (1 and 2) form a resonant triad with a Poincaré mode (3). Graphical construction based on Fig. 1 shows that the conditions (11a) and (11b) can be satisfied. We expand the transverse structure of the Kelvin waves into Fourier series and select transverse modes to satisfy (11c).

For simplicity we take the first-order solution to consist only of the three modes involved in a resonant triad.

$$u^{(0)}(x,y,t)=\sum_{j=1}^{3}\mathscr{U}_j(\tau)e^{i(k_jx+l_jy-\sigma_jt)}+\text{c.c.}, \tag{12}$$

where the slow time $\tau=\alpha t$, has been introduced in anticipation of the resonant growth and c.c. stands for complex conjugate. Although this simple approach will demonstrate the possible instability of the Kelvin modes due to the triad resonant interaction, it must be kept in mind that the Kelvin wave components (1 and 2) are single modes isolated from an infinite Fourier series (in the transverse direction). Thus the expression (12) will never be the full solution of the problem, nor will this resonant triad be the only one possible. In Sec. IV this simplified analysis will be extended to include the full problem and the result compared to numerical solutions.

Substituting (12) into Eq. (6a) we obtain, at $O(\alpha)$,

$$\begin{aligned}\mathscr{L}(u^{(1)})=&\sum_{j=1}^{3}\left[2\sigma_j\left(1+\frac{\beta}{6}k_j^2\right)-k_j\right]\times i\dot{\mathscr{U}}_je^{i(k_jx+l_jy-\sigma_jt)}+\text{c.c.}\\&-\frac{3}{2}\left[\left(\sum_{j=1}^{3}\mathscr{U}_je^{i(k_jx+l_jy-\sigma_jt)}+\text{c.c.}\right)\times\left(\sum_{j=1}^{3}ik_j\mathscr{U}_j\,e^{i(k_jx+l_jy-\sigma_jt)}+\text{c.c.}\right)\right]_t,\end{aligned} \tag{13}$$

where $\dot{\mathscr{U}}_j=d\mathscr{U}_j/d\tau$. In order to avoid secular terms in $u^{(1)}$ we must take

$$\dot{\mathscr{U}}_1 = -\tfrac{3}{2} i\sigma_1 \mathscr{U}_2^* \mathscr{U}_3, \tag{14a}$$

$$\dot{\mathscr{U}}_2 = -\tfrac{3}{2} i\sigma_2 \mathscr{U}_1^* \mathscr{U}_3, \tag{14b}$$

$$\dot{\mathscr{U}}_3 = -\frac{3}{2} i \frac{\sigma_3}{1 + (\Gamma/k_3\sigma_3)(1 + l_3^2)} \mathscr{U}_1 \mathscr{U}_2, \tag{14c}$$

which are the governing equations for the slow time variations of the Fourier components $\mathscr{U}_j$.

Although these equations have well-known analytical solutions (see, for example, Craik[7]) we will only consider simple approximate solutions in order to deduce the stability of the Kelvin modes for two cases of interest.

(i) If only one of the Kelvin wave components is of finite size initially, with the other Kelvin mode and the Poincaré mode of much smaller size, we obtain for small τ

$$\dot{\mathscr{U}}_1 \simeq 0, \tag{15a}$$

$$\ddot{\mathscr{U}}_j + \lambda^2 \mathscr{U}_j = 0 \quad (j = 2,3), \tag{15b}$$

with

$$\lambda^2 = \frac{9}{4} \frac{\sigma_2\sigma_3}{1 + (\Gamma/k_3\sigma_3)(1 + l_3^2)} \mathscr{U}_1^2 > 0.$$

This solution consists of small oscillations about the initial state, showing the finite Kelvin wave component to be neutrally stable. This is consistent with Hasselmann's[8] result, which states that a finite component is neutrally stable with respect to infinitesimal perturbations for the case of difference interaction, $\sigma_1 = \sigma_2 - \sigma_3$, where σ_1 is the frequency of the finite component.

(ii) If both Kelvin wave components are finite initially, with the Poincaré mode of much smaller size, we obtain (valid for small τ)

$$\dot{\mathscr{U}}_j \simeq 0 \quad (j = 1,2), \tag{16a}$$

$$\dot{\mathscr{U}}_3 \simeq \lambda, \tag{16b}$$

where

$$\lambda = -\frac{3}{2} i \frac{\sigma_3}{1 + (\Gamma/k_3\sigma_3)(1 + l_3^2)} \mathscr{U}_1 \mathscr{U}_2$$

is a constant. Also from (14a)–(14c) we obtain $(d/d\tau)(\sigma_2 \mathscr{U}_1 \mathscr{U}_1^* - \sigma_1 \mathscr{U}_2 \mathscr{U}_2^*) = 0$. Thus the Poincaré mode grows linearly in time at the expense of both Kelvin modes, showing them to be unstable.

IV. NUMERICAL SOLUTIONS

As discussed above, expression (12) is not the full solution of Eqs. (4a) and (4b) with the initial conditions consisting of linear Kelvin waves, but rather serves the purpose of demonstrating the possible instability due to resonant triads.

To verify the resonance process in the full problem we solve Eqs. (4a) and (4b) numerically. In order to extend the analysis carried out in Sec. III to include the full solution of Eqs. (4a) and (4b), it is helpful to look at the solution for v rather than u, since the transverse velocity is associated only with the Poincaré waves and the solution for v thus represents the Poincaré wave field only. Combining Eqs. (4a) and (4b) into one equation in v results in

$$\mathscr{L}(v) = \frac{3}{2}\alpha\left(1 + \frac{\partial}{\partial y}\right)(uu_x), \tag{17}$$

where $\mathscr{L}$ is the linear operator given by (9). In accordance with case (ii) above, we take the initial conditions to consist of two finite Kelvin modes,

$$u(x,y,0) = u_0(x)e^{-y}$$
$$= (\mathscr{U}_1 e^{ik_1x} + \mathscr{U}_2 e^{ik_2x})e^{-y} + \text{c.c.}, \tag{18a}$$

$$v(x,y,0) = 0, \tag{18b}$$

which form a resonant triad with the nth Poincaré mode at $(k_3,\sigma_3) = (k_1 + k_2, \sigma_1 + \sigma_2)$. For small τ, we approximate the right-hand side of (17) by

$$-\frac{3}{2}\alpha\left(1 + \frac{\partial}{\partial y}\right)[uu_x] \simeq \frac{3}{2}\alpha[u_0 u_{0,x}]e^{-2y}, \tag{19}$$

i.e., neglect the Poincaré modes and take the Kelvin modes equal to their initial values.

In order to solve for the resonant growth of v, we write

$$v(x,y,t) = \sum_{n=1}^{\infty} \mathscr{V}_n(\tau) e^{i(k_3x - \sigma_3t)} \sin\left(\frac{n\pi y}{W}\right) + \text{c.c.}, \tag{20}$$

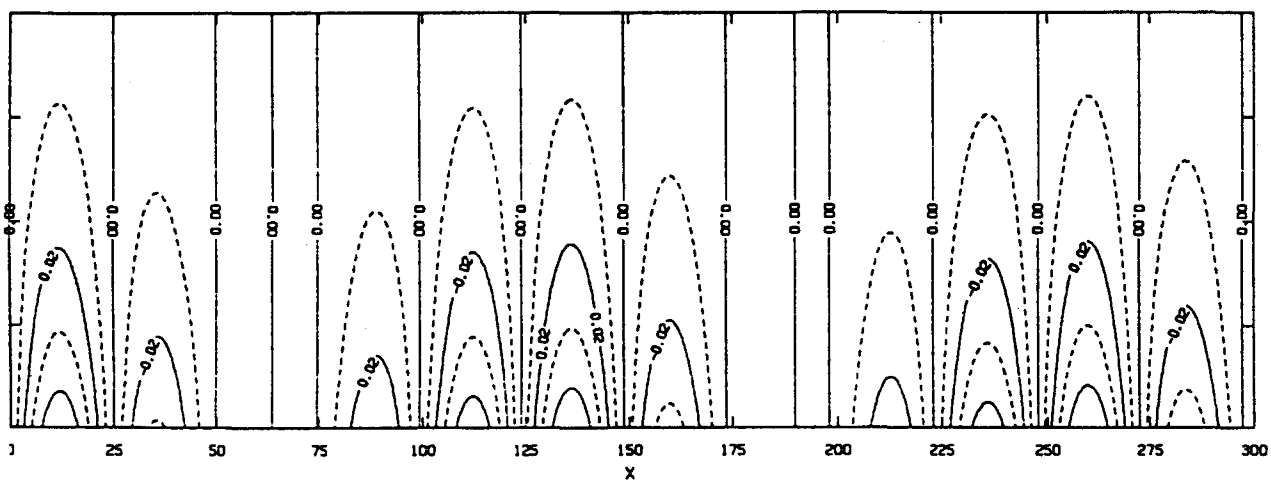

FIG. 2. Contour plot of the initial condition for u, made up of two linear Kelvin modes. Note that only 3/5 of the full length of the channel is shown here.

having neglected all contributions, except from the sum wavenumber $k_3 = k_1 + k_2$, which resonate.

Substituting this into (17), using (18) and (19), and solving for the resonant mode in v gives

$$\mathscr{V}_n(\tau) = \frac{3}{2} i \frac{\mathscr{U}_1 \mathscr{U}_2 b_n}{1 + (\Gamma/k_3\sigma_3)[1 + (n\pi/W)^2]} \times \left(\frac{1}{r} e^{ir\tau} - \frac{1}{r} - i\tau\right), \tag{21a}$$

with

$$r = \frac{k_3}{\alpha[1 + (\beta/6)k_3^2]} \left\{1 + \frac{\Gamma}{k_3\sigma_3}\left[1 + \left(\frac{n\pi}{W}\right)^2\right]\right\}, \tag{21b}$$

where b_n is the Fourier sine coefficient of e^{-2y}. The last term in (21a) represents the secular growth of the Poincaré wave and corresponds to Eq. (16b) for $\mathscr{U}_3$.

To verify this result the coupled evolution equations (4a) and (4b) are solved numerically using the same finite difference scheme described by Melville, Tomasson, and Renouard,[2] but with the boundary conditions

$$u(x = 0,y,t) = (\mathscr{U}_1 e^{-i\sigma_1 t} + \mathscr{U}_2 e^{-i\sigma_2 t})e^{-y} + \text{c.c.}, \tag{22a}$$

$$v(x = 0,y,t) = 0, \tag{22b}$$

and using one-sided, first-order derivatives to express an open boundary condition at $x = L$. The initial conditions are those given by (18a) and (18b).

In the run presented here we used the parameters

$$\mathscr{U}_1/c_0 = \mathscr{U}_2/c_0 = 0.0167,$$
$$(kW)_1 = 5.00, \quad (\sigma/f)_1 = 2.47,$$
$$(kW)_2 = 7.43, \quad (\sigma/f)_2 = 3.63,$$
$$R/W = 0.5, \quad h/W = 0.05, \quad L/W = 12, \tag{23}$$

where the wavenumbers and frequencies are chosen to form a resonant triad with the first Poincaré mode.

Figure 2 shows a contour plot of the initial conditions for u and Fig. 3 shows contour plots of the solution for u and v at a larger time, corresponding, approximately, to the time it takes a Kelvin wave to propagate the length of the channel. In Figs. 2–4 we have renormalized the variables using the scaling

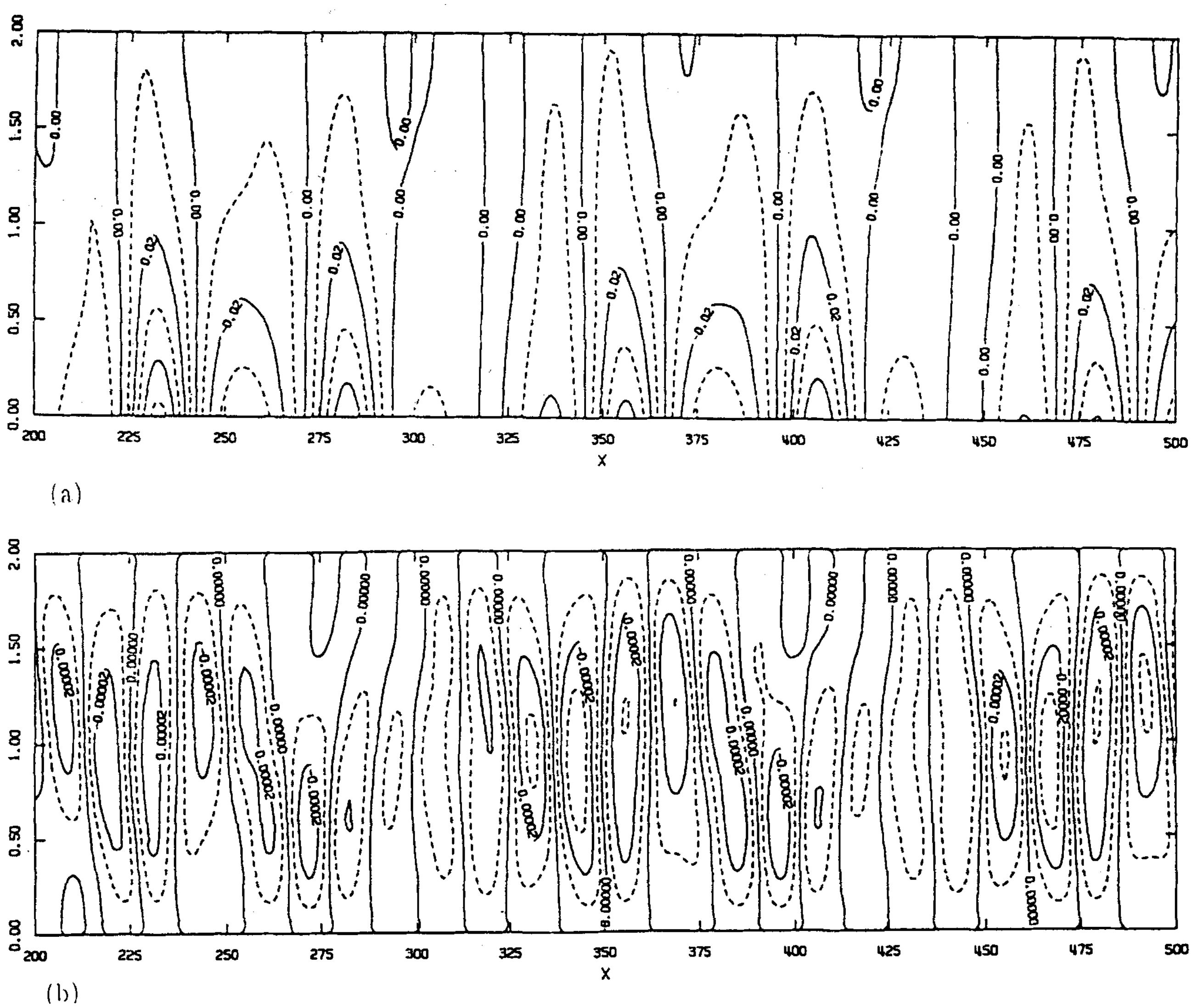

FIG. 3. Contour plots of the solution for (a) u and (b) v at $t = 400$, corresponding approximately to the time a linear Kelvin mode has traveled the length of the channel. Note that only 60% of the full length of the channel is shown here.

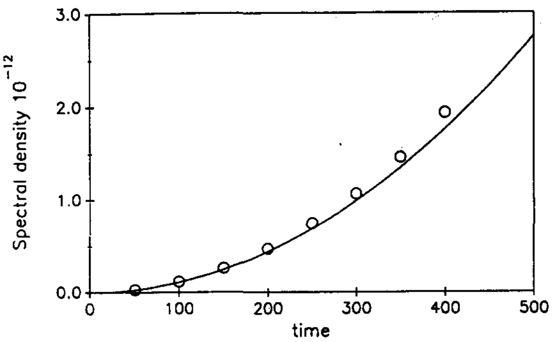

FIG. 4. The growth of the first Poincaré mode, shown as the spectral density of v at the resonant wavenumber k_3: prediction by Eq. (21a) (—) and calculation from the numerical solutions (O).

$$x=\sqrt{\beta/6}\,x',\quad y=y',\quad t=\sqrt{(\beta/6)}\,t',$$
$$u=(2/3\alpha)u',\quad v=\tfrac{4}{3}\sqrt{6}(1/\alpha\Gamma\beta^{1/2})v', \tag{24}$$

where primes denote the new nondimensional variables.

The solution for v is made up of the Poincaré modes only, since no transverse velocity is associated with the Kelvin modes, while the solution for u is made up of both Kelvin and Poincaré modes. (The potential vorticity mode is of smaller order and can be neglected.) In the solution for u the result of the nonlinearity is to steepen the crests and flatten the troughs of the Kelvin waves. Also, we notice that as the Kelvin waves propagate down the channel their crests curve slightly backward away from the right-hand wall, together with the decay of the amplitude along the crest becoming less than the linear Kelvin wave decay. This can be attributed to the superposition of the resonated Poincaré waves on the straight crested linear Kelvin waves (cf. Melville, Tomasson, and Renouard[2]). In the solution for v we notice the dominance of the first transverse mode, especially as the waves travel down the channel. This is of no surprise since we chose the wavenumbers of the Kelvin modes to form a resonant triad with the first Poincaré mode.

In Fig. 4 the resulting growth of the first Poincaré mode at the resonant wavenumber k_3 is compared to the resonant growth predicted by Eq. (21a) and (21b). Here we have isolated one group of waves, the groups being quite well defined in the solution for u (cf. Fig. 2 and 3), and follow the group as it propagates down the channel. The predicted growth is calculated according to Eqs. (21a) and (21b), using $\mathscr{U}_1$ and $\mathscr{U}_2$ from the initial conditions. The observed growth is obtained from the numerical solution for v by first taking the Fourier sine transform in y, and then Fourier transforming the first transverse mode in x, using a fast Fourier transform (FFT) algorithm. The agreement between predicted and observed growth is very good for all t, thus confirming the resonant growth of the Poincaré mode at the expense of the Kelvin modes.

TABLE I. Parameters estimated for the Strait of Gibraltar.

Upper layer depth, h_1	100 m
Lower layer depth, h_2	400 m
Normalized density difference, $\Delta\rho/\rho$	0.002
Linear phase speed, $c_0=\left(g\frac{\Delta\rho}{\rho}\frac{h_1h_2}{h_1+h_2}\right)^{1/2}$	1.25 msec^{-1}
Rossby radius, c_0/f	15 km
Approximate channel width, W	20 km
Approximate channel length, L	40–80 km
Wave amplitude, $a_1=a_2$	2.2 m

V. DISCUSSION

We have shown that in the scaling considered here, linear Kelvin waves in a channel can be unstable due to a triad resonant interaction with the linear Poincaré modes of the channel. A result of the instability is a secular growth of the Poincaré mode in time, together with a decay of the Kelvin modes. Associated features are an apparent curvature of the crest of the Kelvin waves, together with the amplitude decaying more slowly along the crest of the waves than for the linear Kelvin waves. Both features can be attributed to the superpositon of the Poincaré waves on the straight crested linear Kelvin waves.

It is of interest to ask if this instability is significant for observed natural flows. The parameters used in the numerical solutions presented in Sec. IV were chosen to correspond approximately to parameters estimated for the Strait of Gibraltar, given in Table I. The fraction of the total energy of the system that is in the first Poincaré mode was found to be proportional to the spectral density shown in Fig. 4 to within an error of 5% to 10%. As the wave group travels the length of the channel the first Poincaré mode gains only about 2.5% of the total energy through the resonance interaction discussed above. However, we note that the nonlinearity used here is very weak, the initial amplitude of each Kelvin mode being only 2.2 m, which is considerably less than the largest internal wave amplitude observed there. Increasing the strength of the nonlinearity [staying within the limits of the scaling given by (3a) and (3b)] would result in a stronger resonant interaction, giving stronger decay of the Kelvin modes together with more visible wavefront curvature. For example, doubling the initial amplitude of each Kelvin mode to 4.4 m, which is still consistent with the scaling (3a) and (3b) and still considerably less than the largest observed amplitudes, quadruples the energy in the Poincaré mode, resulting in the Poincaré wave gaining about 10% of the total energy of the system as the wave group travels the length of the channel.

ACKNOWLEDGMENTS

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