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A MODEL OF THE TWO-PHASE TURBULENT JET

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Abstract – A model of the two-phase turbulent jet is presented. Consideration is given to cases in which the primary fluid phase contains a secondary phase of rigid particles. The mass fraction of the secondary phase is at most of order unity while its volume fraction is much less than unity. A set of model differential equations is developed for cases in which the mean velocities of the phases are sensibly equal. A first-order closure scheme for the axisymmetric jet is devised and the resulting equations solved numerically. This scheme accounts for momentum transfer between the phases and the imperfect response of the particles to the fluid turbulence. Satisfactory agreement with published experimental data is obtained for computed values of the mean velocity, and the mean mass flux of the particles.

NOMENCLATURE

a, particle radius;
C_g, constant, see equation (3.25);
C_v, constant, see equation (3.6);
d, particle diameter;
D, nozzle diameter;
F_{pi}, force of interaction between the phases;
g, gravitational acceleration;
G, mean mass flux of secondary fluid;
l, length scale of energetic turbulence;
p, pressure;
r, radial coordinate;
S_{ij}, stress in the secondary fluid;
S_p, turbulent Schmidt number of secondary fluid (ν_p/κ_p);
S'_p, turbulent Schmidt number (ν_f/κ_p);
t_*, Stokesian relaxation time;
t_e, time scale of energetic primary fluid turbulence;
T_L, Lagrangian integral time scale;
u, velocity scale of energetic turbulence;
u_i, velocity of primary fluid;
U_x, mean axial velocity of primary fluid;
v_i, velocity of secondary fluid;
x, axial coordinate;
x_i, Cartesian coordinates.

Greek symbols

δ, mean velocity half-radius;
δ_g, mean mass flux half-radius;
η, Kolmogorov length scale;
κ_p, eddy diffusivity of secondary fluid;
Λ_g, Eulerian integral length scales;
μ, viscosity of primary fluid;
ν, kinematic viscosity of primary fluid;
ν_f, eddy viscosity of primary fluid;
ν_p, eddy viscosity of secondary fluid;
ρ, density of primary fluid;
ρ_p, density of secondary fluid;
ρ_s, particle material density;
τ_{ij}, viscous stress tensor;
Γ, Reynolds stress of primary fluid;
ϕ, azimuthal coordinate;
χ, ρ_p/ρ.

Subscripts

0, jet exit condition;
m, jet axis condition.

Superscripts

0, value of primary fluid variable in corresponding clean flow;
$'$, turbulent fluctuation.

1. INTRODUCTION

IN A COMPANION study (Melville and Bray [1]) we found that some of the more important features of the two-phase turbulent jet could be correlated with the single phase jet by functions of the particle loading ρ_{po}/ρ, where ρ_{po} is the density of the particle field at the nozzle and ρ is the density of the incompressible suspending fluid. It was apparent however that the simple description of the mass and momentum transfer processes given there offered little understanding of the effects of the particles on the fluid turbulence and the response of the particles to that turbulence. It became clear that a more detailed investigation was required, one which modelled these processes, taking into account the characteristics of the particles and the turbulence, and tested them against the empirical measurements. To our knowledge this had not been done before for the jet flow although various workers had speculated as to how some of these processes might be modelled.

Owen [2] argued that in a shear flow, in which the assumption of local equilibrium was applicable, the particles' presence would lead to an increase in the energy dissipation in the ratio $(1+\bar{\rho}_p/\rho)$ for $t_* \ll t_e$, where $\bar{\rho}_p$ is the local mean particle mass density, and t_* and t_e are respectively the response time of a

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typical particle and the characteristic time of the energetic turbulence. This is in substantial agreement with the conclusions of Kuchanov and Levich [3] for a homogeneous, isotropic turbulence. Owen went on to argue that with a given mean velocity profile the turbulent velocity scale u and eddy viscosity are decreased in the ratio $(1+\bar{\rho}_p/\rho)^{-1/2}$. For $t_* \gtrsim t_e$ this ratio was given to be $\{1+(\bar{\rho}_p/\rho)(t_e/t_*)\}^{-1/2}$. Owen noted that the quantity controlling the interaction between the particles and fluid was the relaxation time t_*. He concluded that in the jet, for $t_* \ll t_e$, the axial force on the fluid from the particles arose from the transport of the particles across the mean shear by their radial turbulent velocity. We shall see in Section 2 that this is only one component of the force, and that other important contributions were neglected.

Abramovich [4, 5] has attempted to apply a form of the mixing length theory to determine the turbulence shear stress in the jet. He found that the ratio of the stress in the particle-laden flow to that in the clean flow is

$$\Gamma/\Gamma_0 = (1+\bar{\rho}_p/\rho)^{-1}, \tag{1.1}$$

but the argument he used to obtain this result is inconsistent (see Melville [6]).

The mass transfer of particles in turbulent flows is not well understood especially in cases of significant particle loading, $\rho_p/\rho \gtrsim 1$. In dilute developed jet flows, $\rho_p/\rho \ll 1$, it is found that the mean density profile becomes similar to the mean velocity profile, and that to a first approximation the particle transport may be described by a turbulent Schmidt number, S'_p, which depends on the particle and flow characteristics. As t_*/t_e tends to zero there appears to exist a limiting value of S'_p which is not greatly different from the corresponding Schmidt number for mass transport in the single phase flow (see Goldschmidt *et al.* [7]). The experimental measurements show a decrease in S'_p with increasing particle size, and hence response time t_*. Hinze [8] has suggested that this is due to particle size effects. He argued that the diffusivity increase is due mainly to an increase in the Lagrangian integral length scale of the particle velocity, but the effect is not very strong in the axisymmetric jet flows.

Ideas such as these have not previously been incorporated into a mathematical model and tested against the experimental measurements of two-phase jets containing a significant loading of particles. We are particularly concerned with cases in which ρ_p/ρ is at most of order unity, and the volume fraction of particles is much less than unity. In Section 2 is developed a set of Reynolds averaged model equations which we believe are capable of representing the significant linear momentum exchange processes. A first-order closure scheme for the mean velocity and density fields is derived in Section 3. It is argued that a more sophisticated scheme is unjustified at present. The equations are solved numerically and tested against published experimental data, and the results presented in Section 4. A comprehensive discussion is presented in Section 5, with emphasis being placed on the development of the model and the directions it suggests for future experiments.

2. THE MODEL DIFFERENTIAL EQUATIONS

2.1. *The time-dependent governing equations*

There are perhaps two main approaches to the modelling of two-phase flows. The first considers the suspension to be represented as a single inhomogeneous continuum. The forces of interaction between the phases then give rise to internal stresses of the composite fluid which must be related to the bulk variables (e.g. velocity) by constitutive relations. An example of this approach is given by Barenblatt [9]. The other approach is to maintain the identity of the phases and assume that continuous bulk variables of each phase may be defined by appropriate spatial and temporal averages. The interaction between the phases in this case results in explicit body forces in the momentum equations. The advantages of each method depend on the problem being studied, available methods of solution, and the experimental measurements. In modelling turbulent flows closure hypotheses must be made and tested. The experimental measurements are then of crucial importance and a theory which relates closely to the variables measured has a considerable advantage over one which may be tested only indirectly. For this reason we chose to model the suspension as two interpenetrating continua and employ the equations derived by Marble [10].

Marble based his formulation on the collisionless Boltzmann equation for the evolution of the distribution function describing the particle cloud. Moments of that equation lead to the mass and momentum conservation equations for the secondary fluid comprised of the particles:

$$\frac{\partial \rho_p}{\partial t} + \frac{\partial}{\partial x_j}(\rho_p v_j) = 0 \tag{2.1}$$

and

$$\rho_p \frac{\partial v_i}{\partial t} + \rho_p v_j \frac{\partial v_i}{\partial x_j} = -F_{pi} + \frac{\partial}{\partial x_j} S_{ij}, \tag{2.2}$$

where ρ_p is the mass of particles per unit volume of the mixture. Thus $\rho_p = \rho_s \theta$, where ρ_s is the mass density of the particle material and θ is the volume fraction of particles; v_i is the velocity of the secondary fluid such that $\rho_p v_i$ is the ith component of the particle momentum per unit volume of the mixture; F_{pi} is the force on the primary fluid per unit volume due to interaction with the particles; S_{ij} is the stress tensor due to particle motion at velocities different from the local mean v_i.

For the cases of interest the volume fraction of particles is negligible, typically $O(10^{-3})$, thus the density of the primary fluid is sensibly equal to the material density of the suspending fluid. With this assumption the mass and momentum conservation equations for the primary fluid become (Marble [10]):

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_j}(\rho u_j)=0 \tag{2.3}$$

and

$$\rho\frac{\partial u_i}{\partial t}+\rho u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{\partial \tau_{ij}}{\partial x_j}+F_{pi}, \tag{2.4}$$

where ρ, u_i and p are the local density, velocity and pressure respectively and τ_{ij} is the viscous stress tensor. Since $\rho/\rho_s \ll 1$ virtual added mass effects have been neglected.

The above equations are for isothermal flows. Marble considered the more general case and proposed energy equations which are not given here.

The stress tensor S_{ij} in equation (2.2) is analogous to the kinetic theory description of τ_{ij}, arising as a result of particles in an elementary volume having finite peculiar velocities. If all the particles in the element have the same velocity S_{ij} is zero. We shall assume this to be the case. This assumption naturally leads to a consideration of the validity of the continuum hypothesis. If the particle number density is small the smallest elementary volume required for meaningful statistical averages may be large enough to encompass significant inhomogeneities of the primary fluid velocity field. If the particles are responsive to these scales the assumption that S_{ij} is negligible may break down. Under such conditions it is likely that the continuous variables of the primary fluid would need to be redefined on the same scale as those of the secondary fluid. According to Hinze [8] heavy restrictions are required for the continuum concept to be valid. He suggests that these restrictions require the separation distance between particles to be at least an order of magnitude less than the Kolmogorov length scale η. This criterion is arbitrary. To our knowledge such criteria have not been generally resolved. The usual procedure, in a particular case, has been to assume the continuum hypothesis valid and then search for contradictory evidence in comparing the theoretical predictions with the experimental results. This has been our approach.

2.2. *Reynolds averaged equations for thin shear flows*

The specification of the force of interaction F_{pi} is quite general and in order to proceed it must be further defined. We shall assume that it is of the same order of magnitude as the Stokes drag,

$$F_{pi}\sim \rho_p(v_i-u_i)/t_*, \tag{2.5}$$

where

$$t_*=\frac{d^2}{36\nu}(2\rho_s/\rho+1)\approx\frac{d^2\rho_s}{18\nu\rho}.$$

d is the characteristic particle diameter, and ρ_s is the mass density of the secondary phase material. Further, both ρ_p/ρ and t_*/t_e are assumed to be of order unity. With these restrictions it is shown in the Appendix that the mean velocity of the secondary fluid is approximately equal to that of the primary fluid in the jet. It is then redundant to retain a separate momentum equation for the second fluid. Using equation (2.2) to eliminate F_{pi} from equation (2.4), transforming to cylindrical co-ordinates, assuming ρ constant and employing the thin shear layer approximations, equations (2.1–2.4) become

$$\frac{\partial U_x}{\partial x}+\frac{1}{r}\frac{\partial}{\partial r}(rU_r)=0, \tag{2.6}$$

$$U_x\frac{\partial \bar\rho_p}{\partial x}+U_r\frac{\partial \bar\rho_p}{\partial r}=-\frac{1}{r}\frac{\partial}{\partial r}(r\overline{\rho'_p v'_r}), \tag{2.7}$$

$$\begin{aligned}\rho\left[U_x\frac{\partial U_x}{\partial x}+U_r\frac{\partial U_x}{\partial r}\right]&=-\frac{\partial P_0}{\partial x}-\frac{\rho}{r}\frac{\partial}{\partial r}(r\overline{u'_xu'_r})\\&-\frac{\rho}{\rho+\bar\rho_p}\left[\frac{\bar\rho_p}{\rho}\left(-\frac{\partial P_0}{\partial x}-\frac{\rho}{r}\frac{\partial}{\partial r}(r\overline{u'_xu'_r})\right)\right.\\&\left.+\overline{\rho'_pv'_r}\frac{\partial U_x}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}(r\bar\rho_p\overline{v'_xv'_r})\right],\end{aligned} \tag{2.8}$$

(see Melville [6] for details). Upper case and overbarred letters refer to mean quantities; x and r denote the axial and radial components respectively, and P_0 is the free stream pressure. Equations (2.6–2.8) are three equations for the three mean fields $\bar\rho_p$, U_x, U_r. Equations (2.6) and (2.7) are familiar forms of the mass conservation equations for the primary and secondary fluids respectively. Equation (2.8) is the axial momentum conservation equation for the primary fluid. The last term on the RHS of that equation represents the mean force on the fluid due to the particles. This term is clearly zero if $\rho_p=0$ and the equation is then the usual axial momentum equation for the single phase jet.

Owen [2] considered the force on the fluid due to the particles in an axisymmetric jet under the condition that $t_*\ll t_e$. This was taken to imply that the particles are fully responsive to the turbulent fluctuations. Owen considered that the axial acceleration of the particle was due only to the turbulent advection across the mean velocity field leading him to conclude that the total average axial force on a unit volume of fluid is

$$\bar F_{px}=-\overline{\rho'_pv'_r}\frac{\partial U_x}{\partial r}. \tag{2.9}$$

Under the same approximation, neglecting the axial mean pressure gradient, we have from equation (2.8) that

$$\bar F_{px}=-\frac{\rho}{\rho+\bar\rho_p}\left[\overline{\rho'_pu'_r}\frac{\partial U_x}{\partial r}+\overline{u'_xu'_r}\frac{\partial\bar\rho_p}{\partial r}\right]. \tag{2.10}$$

The discrepancy between (2.9) and (2.10) arises from Owen's incomplete specification of the particles' acceleration.

3. THE CLOSURE SCHEME

3.1. *Introduction*

The particular flow we wish to consider is the axisymmetric two-phase jet issuing into a quiescent fluid having the same properties as the primary fluid of the jet. We wish to obtain solutions for U_x, U_r and $\bar{\rho}_p$, and shall restrict the model to the set of equations (2.6–2.8). That is, we shall not employ any higher order equations for the quantities $\overline{u'_x u'_r}$, $\overline{v'_x v'_r}$ and $\overline{\rho'_p v'_r}$. This is an important decision and requires some justification.

If higher order equations, for $\overline{u'_x u'_r}$ say, were to be used, starting profiles of the relevant quantities would be required for the numerical solution procedure. Such profiles are usually obtained from direct measurement, or by relating such quantities to the mean fields through eddy viscosity or mixing length hypotheses [11]. The second procedure necessarily assumes that such relationships are good first-order approximations. To our knowledge such tested relationships do not yet exist for the two-phase jet: it is our purpose to attempt to devise them. Harsha [11], in an extensive survey and testing of a number of models for free turbulent flows, concluded that the higher order closure schemes, using the kinetic energy equation, only give good results if accurate initial profiles are known. The corresponding profiles are not known for the two-phase jet. The evidence suggested that the best course was to attempt to derive a first-order closure scheme, directly relating the turbulence quantities to the mean fields.

From the evidence reviewed by Melville and Bray [1] there can be little doubt that, in the limit as ρ_p/ρ goes to zero, the mean velocity field tends to that of the corresponding clean jet. Hence our closure scheme for $\overline{u'_x u'_r}$ must tend to that for the clean jet as ρ_p/ρ tends to zero. The usual first-order closure hypothesis is that the mean transfer of momentum by the Reynolds stress can be described by a scalar eddy viscosity, ν_f:

$$\overline{u'_x u'_r} = -\nu_f(x, r)\frac{\partial U_x}{\partial r}. \tag{3.1}$$

The practice of relating the Reynolds stress to the mean velocity field in this manner has been severely criticised by some workers, and it should not be used without caution. In the single-phase flow the use of an eddy viscosity is usually justified on the basis of an approximate local equilibrium. While local equilibrium does not strictly apply in the clean axisymmetric jet, it has been found, both experimentally [12] and through theoretical modelling [11], that the eddy viscosity hypothesis is for some purposes a satisfactory representation of the mean momentum transport in such flows. If the addition of the second phase caused the primary fluid turbulence to adjust more slowly to the mean velocity field, or if it introduced additional mechanisms for generation of primary fluid turbulence, we would expect (3.1) to prove a poor approximation. With our restriction to cases in which $V_i \simeq U_i$, $\rho_p/\rho \sim O(1)$ and $t_*/t_e \sim O(1)$ we expect neither of these effects. Thus we assume (3.1) and make the corresponding assumption for the form of $\overline{v'_x v'_r}$, setting

$$\overline{v'_x v'_r} = -\nu_p(x, r)\frac{\partial U_x}{\partial r}. \tag{3.2}$$

$\nu_p(x, r)$ is the eddy viscosity of the turbulent flow of the secondary fluid. Similarly the turbulence mass flux $\overline{\rho'_p v'_r}$ is assumed to be of the form

$$\overline{\rho'_p v'_r} = -\kappa_p(x, r)\frac{\partial \bar{\rho}_p}{\partial r} \tag{3.3}$$

where $\kappa_p(x, r)$ is the eddy diffusivity of the secondary fluid.

With the equations (3.1)–(3.3), and some rearrangement, equations (2.7) and (2.8) become

$$U_x\frac{\partial \bar{\rho}_p}{\partial x} + U_r\frac{\partial \bar{\rho}_p}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\kappa_p\frac{\partial \bar{\rho}_p}{\partial r}\right), \tag{3.4}$$

$$\begin{aligned}\rho\left(U_x\frac{\partial U_x}{\partial x} + U_r\frac{\partial U_x}{\partial r}\right) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r[1+(\bar{\rho}_p\nu_p)/(\rho\nu_f)][1+\bar{\rho}_p/\rho]^{-1}\rho\nu_f\frac{\partial U_x}{\partial r}\right)\\ &\quad + [1+\bar{\rho}_p/\rho]^{-1}\{\kappa_p + [1+(\bar{\rho}_p\nu_p)/(\rho\nu_f)]\\ &\quad \times [1+\bar{\rho}_p/\rho]^{-1}\nu_f\}\frac{\partial \bar{\rho}_p}{\partial r}\frac{\partial U_x}{\partial r}. \end{aligned} \tag{3.5}$$

3.2. *Detailed modelling of turbulent transport coefficients modelling of ν_f*

The modelling of ν_f is based on the eddy viscosity model for the corresponding clean flow. The eddy viscosity ν_f^0 of the clean axisymmetric jet is, to a good first approximation, constant and may be written

$$\nu_f^0 = C_\nu U_m \delta, \tag{3.6}$$

where C_ν is a constant, U_m the centreline mean velocity, and δ the velocity half-radius. Strictly it is not ν_f^0 but, $\nu_{f\gamma}^0$ defined by

$$\nu_{f\gamma}^0 = \overline{u'_x u'_r}\bigg/\left[\gamma\frac{\partial(U_x/\gamma)}{\partial r}\right], \tag{3.7}$$

(where γ is the boundary intermittency factor) which is constant, however we shall use (3.6) at the expense of introducing errors in U_x near the flow boundary.

In Section 1 we introduce the estimate

$$\nu_f/\nu_f^0 \sim (1+\bar{\rho}_p/\rho)^{-1/2}, \tag{3.8}$$

which was obtained by Owen [2] for cases in which the mean velocity profile was given (i.e. the same for both the two-phase flow and the corresponding clean flow), and t_*/t_e was either much less than, or of order, unity. Owen also proposed that for $t_*/t_e \geqslant 1$

$$\nu_f/\nu_f^0 \sim [1+(\bar{\rho}_p/\rho)\cdot(t_e/t_*)]^{-1/2}. \tag{3.9}$$

Treating the estimate (3.8) as an equality and using (3.6) gives

$$\nu_f = \nu_f^0(1+\bar{\rho}_p/\rho)^{-1/2} = C_\nu(1+\bar{\rho}_p/\rho)^{-1/2}U_m\delta, \tag{3.10}$$

which is one of the models tested, and will be referred to as model A. While (t_*/t_e) ranged up to values of $O(10)$ in the flows modelled the results obtained with equation (3.10) demonstrated that the use of (3.9) was not justified.

Abramovich's model (see Section 1),

$$\Gamma/\Gamma^0 = (1+\bar{\rho}_p/\rho)^{-1}, \tag{3.11}$$

under the assumption that the mean velocity profile is given and the eddy viscosity hypothesis is valid becomes

$$\nu_f/\nu_f^0 = (1+\bar{\rho}_p/\rho)^{-1}. \tag{3.12}$$

Despite Abramovich's erroneous derivation of (3.11) the model (3.12) was tested, and is referred to as model B. We also tested the model

$$\nu_f/\nu_f^0 = (1+\bar{\rho}_{pm}/\rho)^{-1}, \quad \bar{\rho}_{pm} = \bar{\rho}_p(x,0), \tag{3.13}$$

which is a simplification of B, and is referred to as model C.

Modelling of ν_p. In the limit as $t_*/t_e \to 0$ we expect ν_p to tend to ν_f; but for finite t_*/t_e we assume that ν_p may be given by an equation of the form

$$\nu_p = \nu_f \cdot \mathscr{h}(t_*/t_e), \tag{3.14}$$

$$\mathscr{h}(t_*/t_e) \to 1 \quad \text{as} \quad t_*/t_e \to 0.$$

Harsha [11] has found that in the clean axisymmetric jet an empirical correlation of the following form holds:

$$\overline{u'_x u'_r} = a_f k_f, \tag{3.15}$$

where

$$k_f = \tfrac{1}{2}\overline{u_i'^2}.$$

This relationship was first suggested by Townsend [13]. a_f is approximately constant over a significant region of the jet and tends to zero at the axis and the boundary. We assume that a similar relationship holds for the secondary fluid:

$$\overline{v'_x v'_r} = a_p k_p,$$

where

$$k_p = \tfrac{1}{2}\overline{v_i'^2}. \tag{3.16}$$

Of course (3.15) and (3.16) are only of use if a_f and a_p are well behaved. We assume them to be equal. Thus

$$\nu_p/\nu_f = \overline{v'_x v'_r}/\overline{u'_x u'_r} = k_p/k_f, \tag{3.17}$$

so from (3.14),

$$\mathscr{h}(t_*/t_e) = k_p/k_f. \tag{3.18}$$

To our knowledge there are no theoretical results relating to the relative turbulent intensity in two phase shear flows. However, there is an expression for k_p/k_f for homogeneous isotropic turbulence [14]:

$$k_p/k_f = 1/(1+t_*/T_L), \tag{3.19}$$

where T_L is the (longitudinal) Lagrangian integral time scale of the primary fluid.

The particle response function on which (3.19) is based is essentially due to the Stokesian drag [15]. Corrsin [16] estimated T_L by

$$T_L = \Lambda/(\overline{u'^2})^{1/2} \tag{3.20}$$

where Λ is the Eulerian spatial integral scale. This estimate is strongly supported by the empirical data of Snyder and Lumley [17].

We assume that equation (3.19) holds for the jet flow. This is a crude assumption but appears to be the best possible at present. It should be used cautiously remembering that it is a result which applies strictly to an equilibrium condition in which the statistics of the velocity field sampled by the particle are stationary. In a developing shear flow the velocity field sampled by the particle is not stationary and the particle is in general not in equilibrium with the surrounding packet of fluid. As t_* increases we expect the departure from equilibrium to increase. However, if t_* is significantly less than the time scale for appreciable development of the flow then an approximate equilibrium should exist.

Substituting (3.19) into (3.17) we have

$$\nu_p = \nu_f(1+t_*/T_L)^{-1}. \tag{3.21}$$

There is some ambiguity in using (3.20) for an anisotropic flow. We have used the following estimates

$$T_L \sim \Lambda_g/(\overline{u'^2})^{1/2}, \tag{3.22}$$

but

$$\nu_f \sim (\overline{u'^2})^{1/2}\Lambda_g,$$

hence

$$T_L \sim \Lambda_g^2/\nu_f \tag{3.23}$$

where Λ_g is the integral scale associated with the correlation $\overline{u'_x(x,r)u'_x(x,r+\Delta r)}$. Now we take T_L to be given by

$$T_L = \Lambda_g^2/\nu_{fm}, \tag{3.24}$$

where ν_{fm} is the eddy viscosity at the axis. T_L is of course comparable to t_e, the time scale of the energetic eddies. Λ_g is assumed constant across the flow and equal to the corresponding value in the clean jet for the same mean velocity profile. We took this to be the mixing length given by Launder and Spalding [18] as 0.075 times the half-width $(= 2.6\delta)$ of the jet. Thus

$$\Lambda_g = 0.075(2.6\delta) = C_g(2.6\delta) \text{ say.} \tag{3.25}$$

This value of Λ_g differs by only 6% from the value of the integral scale measured on the axis by Wygnanski and Fielder [12]. Because of our assumption that Λ_g is constant across any section of the jet (while in general it varies), and approximately equal to its value at the axis, it would not have been consistent to allow a radial variation of T_L with ν_f.

The latter was therefore fixed at its axis value, v_{f_m}, when calculating T_L.

Modelling of κ_p. The turbulent Schmidt number defined by

$$S_p = v_p/\kappa_p \tag{3.26}$$

is assumed constant. In the far field of the jet t_*/T_L tends to zero, so from (3.21) we have that v_p tends to v_f, and

$$S_p \rightarrow S_p' = v_f/\kappa_p. \tag{3.27}$$

The experimental evidence for lightly loaded axisymmetric jets [19, 7] shows that S_p' is not strongly dependent on t_*. For example, Singamsetti's measurements showed that S_p' ranged from 0.85 to 0.69 while t_* increased by two orders of magnitude. We expect that as t_*/t_e, and ρ_p/ρ tend to zero the particles will be transported as a passive contaminant. The turbulent Schmidt number for passive contaminant transport in the clean jet is 0.7 [18]. Thus to satisfy the anticipated limiting behaviour we set

$$S_p = 0.7. \tag{3.28}$$

Equations (3.21), (3.26) and (3.28), will lead to an increase of S_p' with increasing t_*, whereas in Section 1 we reported that empirical data in lightly loaded flows show a decrease in S_p'. Hinze [8] has argued that this behaviour arises when the typical linear particle dimension, d say, is comparable to the dissipation length scale (Taylor microscale, λ). We shall only be considering cases in which $d/\lambda \ll 1$.

4. MODEL RESULTS

The only suitable experiments with which to compare the models in the range of interest [$\rho_p/\rho \sim O(1)$, $t_*/t_e \sim O(1)$] are those of Laats and Frishman [20, 21]. We have reviewed these experiments in some detail elsewhere [1] and shall give only a brief description here. The measurements were carried out using various mixtures of air and corundum powders having characteristic diameters of 17, 32, 49, 72 and 80 μm. The characteristic particle diameter was not defined, but the size distributions which were presented support our assumption that it was the median size. The initial loadings were in the range $0 \leqslant \bar{\rho}_{po}/\rho \leqslant 1.4$, with axial velocities at the nozzle varying from 29 to 60 m/s. The nozzle diameter was 0.035 m. Laats and Frishman's measurements display maxima in some of the centre-line mass flux curves immediately downstream of the nozzle. They attributed this to Magnus forces producing a radial transport of the particles. We do not find this explanation entirely satisfactory [1] and the processes which may account for these phenomena are not included in the equations developed above. Nevertheless, we believe that the equations may represent the processes occurring downstream of these maxima. Thus in testing the closure schemes we have started the calculations in this region.

Solutions of the equations (2.6), (3.4) and (3.5) were obtained from given initial profiles using the numerical procedure of Patankar and Spalding [22]. The details of this procedure are given in [6] along with the method of setting up the initial profiles. The experimental measurements of U_{om}/U_m, G_m/G_{om}, δ/D and δ_g/D vs x/D, the dimensionless streamwise coordinate, were compared with the computed curves. $U_m = U_x(x, 0)$, and G_m is the mean particle mass flux on the axis, which for the thin shear layer approximations is given by

$$G_m = G(x, 0) = \rho_p(x, 0), \; V_x(x, 0).$$

U_{om} and G_{om} are the corresponding quantities at the nozzle. δ and δ_g are the half radius of the axial mean velocity and particle mass flux fields respectively. In referring to the flow conditions we shall use the notation (d, χ_0, U_{om}) where d is the characteristic particle size in μm, $\chi_0 = \bar{\rho}_{po}/\rho$, and U_{om} is in m/s. The empirical constants are $S_p = 0.7$, $C_\iota = 0.028$ and $C_g = 0.075$, except where stated to the contrary. Laats and Frishman did not give U_{om} for each experiment, saying only that it ranged between 29 and 60 m/s. We have calculated U_x and G for U_{om} equal to 29 and 60 m/s, spanning the range cited.

Initial tests of the models showed that none of the closure schemes gave satisfactory agreement with the experiments if t_* was set to zero. These results demonstrated the importance of the particle response time even for the smallest particles (d = 17 μm). The errors arose from too rapid radial transport of the particles, a trend which was to be corrected by incorporating the dependence of v_p on t_* given by equation (3.21). This equation was subsequently used in obtaining all the results presented below. In each case the response time t_* was based on the characteristic diameter cited by Laats and Frishman. This approximation is discussed below. Model C, equation (3.13), was tested and found to compare poorly with the measurements. Model B, equation (3.12), was also tested and for the lighter loadings, $\chi_0 = 0.3$, 0.56, was found to compare favourably with A, equation (3.10). For the heavier loadings the differences were significant, with A giving superior agreement with the data. All subsequent computations incorporated model A.

Figures 1–3 show the predicted axial development of the jet for a representative sample of the cases measured by Laats and Frishman. The agreement between the model and the data is generally satisfactory and the discrepancies are comparable to our estimates of the likely errors of 5–10% in the measurements. Nevertheless, there are some points inviting comment. Figure 1 shows an initial over prediction of δ_g. This may be due to the ratio v_p/v_f being too large in this region, but this effect did not occur for any other set of conditions. It is also possible that an estimate used in setting up the initial profiles was in error [6]. The predictions of U_{om}/U_m and δ/D are generally good and show little dependence on U_{om}. This is not the case for G_m/G_{om}

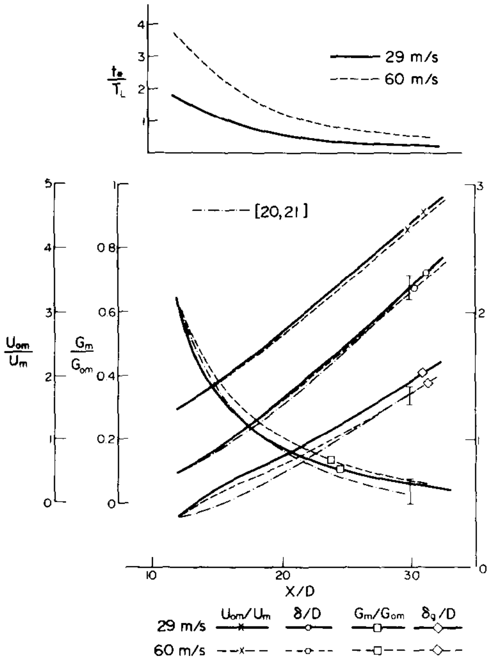

FIG. 1. Comparison of the computed curves using model A with the measured curves of Laats and Frishman [20, 21] for $d = 17\,\mu$m, $\chi_0 = 0.3$, $U_{om} = 29, 60$ m/s, i.e. (17/0.3/29), (17/0.3/60).

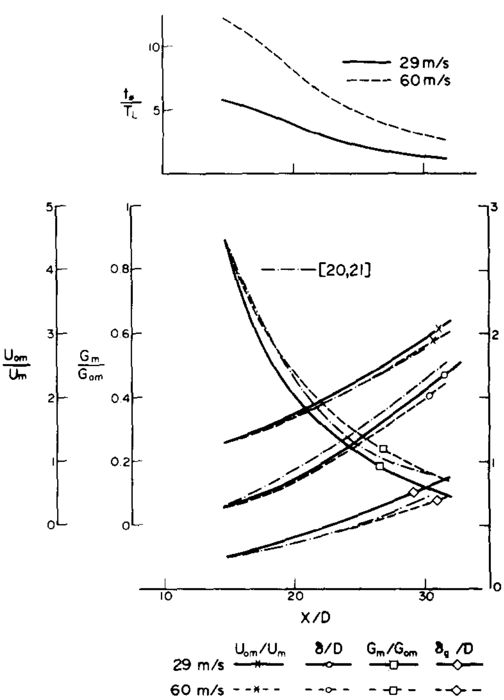

FIG. 2. Model A, (32/0.77/29), (32/0.77/60).

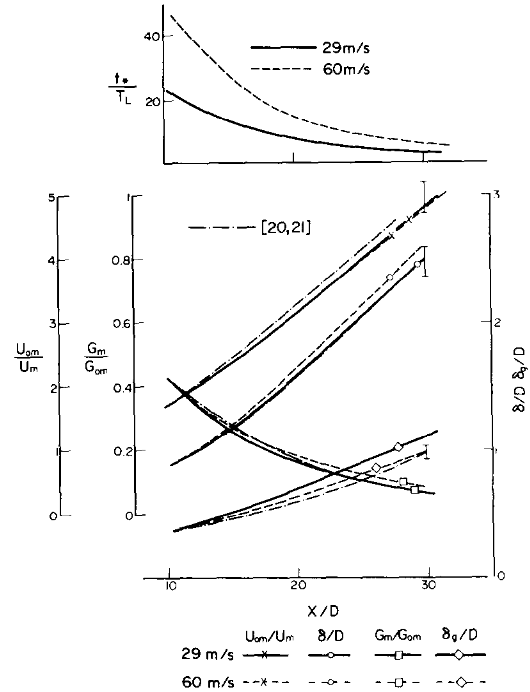

FIG. 3. Model A, (72/0.3/29), (72/0.3/60).

and δ_g/D, in contrast to Laats and Frishman's report that they found G_m/G_{om} independent of U_{om}. Unfortunately their papers do not provide sufficient detail for us to explore this discrepancy. It is possible that the discrepancy is apparent rather than real. Our estimate of the likely errors in the measurements are comparable to the predicted differences in G_m/G_{om} and δ_g/D as U_{om} is varied from 29 m/s to 60 m/s. Thus it is possible that Laats and Frishman were unable to measure this effect. If the discrepancy is real it is likely that it arises as a result of the dependence of T_L (and hence t_*/T_L) on U_{om}.

One of the notable features of the model is that it gives satisfactory agreement in cases where t_*/t_e is $O(10)$ whereas we expected it to be restricted to cases in which $t_*/t_e \sim O(1)$. In fact the conditions which best agree with the experimental measurements are (72/0.3/29) and (72/0.3/60) in Fig. 3 where t_*/t_L is as much as 40.

Figure 4 shows representative computed mean velocity and mass flux profiles. Laats and Frishman reported that the velocity profile of U/U_m approached that of the clean jet as x/D increased. This trend is shown in our results, Fig. 4, with the profiles tending to that for the constant eddy viscosity model of the clean jet. The experimental profiles reported all fell within the hatched region shown. The agreement is good and the slight discrepancy for $r/\delta > 2$ is most likely accounted for by our neglect of the intermittency. Laats and Frishman found that over the range of their experiments the mean particle mass flux profiles were invariant and given by

$$G/G_m = \exp[-0.69(r/\delta_g)^{4/3}]. \qquad (4.1)$$

Figure 4(b) compares our computed profiles against this empirical curve. The differences are clearly significant and it is likely that they arise from two main sources. Firstly, we have modelled the particle size distribution as a monodisperse sample, and secondly, we have not included any radial dependence in our modelling of T_L. In the real flow we would expect the mass flux profiles to be influenced by the particle size distribution and the radial

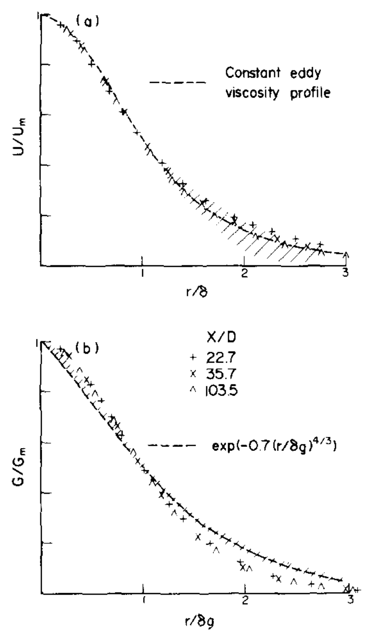

FIG. 4. (a) Mean velocity profiles obtained using model A, (32/0.77/29). Also shown is the theoretical profile for the constant eddy viscosity model of the clean jet. All the measured profiles of Laats and Frishman [20, 21] lay within the hatched region. (b) Mean particle mass flux profiles using model A, (32/0.77/29). Also shown is the curve with which Laats and Frishman [20, 21] correlated their measured profiles. The hatched region shows the scatter of their results.

dependence of the flow structure. The first of these points is briefly considered in the next section.

In model building of this sort one of the aims is to reduce to a minimum the number of empirical constants and wherever possible to avoid introducing new ones. By this means one tries to maintain some degree of generality. It is one of the favourable features of this model that no new constants have been employed. We have only used three S_p, C, and C_g, all of which may be used in a first-order closure scheme for modelling the single phase axisymmetric jet containing a passive contaminant. The value of C, was calculated from the clean jet measurements to be 0.028. We found the results to be very insensitive to changes in C_g. The differences in the predicted mean fields which resulted from increasing S_p from 0.7 to 1.0 were found to be within the estimated errors of the experimental measurements.

The predicted effect of a change in particle diameter d, for fixed conditions, is shown in Fig. 5. The initial conditions are those for (32/0.56/29) but they have been used also with particles of 17 μm and 72 μm dia. The less rapid decay of G_m/G_{om} with increasing particle size is apparent and is due to the less rapid radial transport of the larger particles. This leads to the larger particles transferring momentum to the fluid at a lower rate. This is evident from the decrease in δ/D with increasing particle size; leading to a positioning of the virtual origin of the far field (i.e. complete momentum transfer) further downstream.

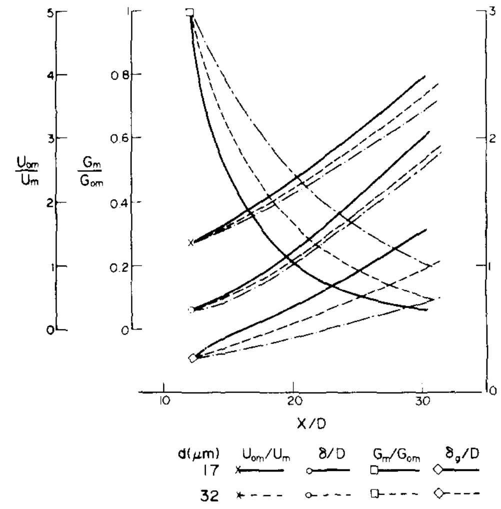

FIG. 5. The effect of changing particle diameter d, for fixed initial conditions. The initial conditions are those for (32/0.56/29) but they have also been used to compute the curves for (17/0.56/29) and (72/0.56/29).

The computed results showed the anticipated behaviour for large x/D, with U_{om}/U_m, $(G_{om}/G_m)^{1/2}$, δ and δ_g becoming linear in x/D as all the momentum is transferred to the primary fluid [1].

5. DISCUSSION

The differential equations and closure scheme presented here have been shown to give satisfactory agreement with experiments in regions of the two-phase jet downstream of the first few orifice diameters. With the reduction of flow time scales near this orifice significant mean velocity lag may be present and the model equations used here would no longer be satisfactory. In addition, the importance of the various possible modes of particle/fluid interaction in the vicinity of the orifice is not clear and no attempt was made to incorporate them in this model. It is likely that shear induced lift and Magnus forces are of some importance.

The closure assumptions employed here are relatively simple. The most important transport coefficient is ν_f; the values of ν_p and κ_p being dependent on it. Of the models for ν_f tested, that based on Owen's result, equation (3.8), gave the best agreement with experiment. Surprisingly it seems to work well outside its expected range of validity i.e. $\rho_p/\rho \sim O(1)$, $t_*/t_e \sim O(1)$. The results for (72/0.3/60) show satisfactory agreement even for $t_*/t_e \sim O(10)$. No experimental data is available for testing the model at density ratios ρ_p/ρ greater than order unity. In view of the performance of model A for $t_*/t_e > O(1)$ we did not test the estimate (3.9) suggested by Owen for this case. The modelling of ν_p and κ_p appears to represent satisfactorily the response of the secondary fluid to the turbulent primary fluid. Our model of the particle mass transport does not agree with a recent one proposed by Abramovich and Girshovich [23]. From a mixing-length argument they arrived at the

Schmidt number relationship

$$S_p' = (2+\bar{\rho}_p/\rho)S_g$$

for particles which faithfully follow the turbulence. S_g is the Schmidt number for a gaseous contaminant in the corresponding single phase flow. According to out model in this case

$$S_p' = S_g.$$

Abramovich and Girshovich's model is not supported by the experimental evidence for lightly loaded jets where it is found that S_p' is approximately equal to S_g [7]. Further, the comparison of their model with the experiments (Fig. 1, [23]) shows only very limited qualitative agreement.

An improvement in our model may be achieved by a more realistic representation of the particle size distribution. Instead of using a single time constant t_* and density ρ_p the distribution could be broken up into a number of size ranges, attributing a time constant and density to each. In principle this extension is no more complex than the present approach and the numerical solution procedure of Patankar and Spalding is capable of solving the additional conservation equations without excessive computation.

Despite the improvements that may be made to this closure scheme it is likely that they will only prove to be marginal and substantial improvement may only come from a higher order closure employing Reynolds stress equations. The development and testing of such models will require further experimental measurements. We believe that these experiments should be mainly concerned with identifying the processes occurring in the vicinity of the nozzle and decoupling particle size effects from those of particle concentration.

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APPENDIX

Transforming equation (2.2) to cylindrical polar coordinates, averaging and using the estimate (2.5), we see that if the difference between the mean axial velocities is to be dynamically significant then $\bar{\rho}_p(V_x - U_x)/t_*$ can at most be of the same order as the term $\rho U_r(\partial U_x/\partial r)$;

i.e. $$\bar{\rho}_p(V_x - U_x)/t_* = O\left(\rho U_r \frac{\partial U_x}{\partial r}\right). \quad \text{(A.1)}$$

In the corresponding clean jet the turbulence time scale t_e is estimated by

$$t_e \sim \left|\frac{\partial U_x}{\partial r}\right|^{-1} \quad \text{(A.2)}$$

and we expect that for the particle loadings we are considering this estimate is still appropriate. Now we have assumed that

and $$\left.\begin{aligned} t_* &= O(t_e) \\ \rho_p &= O(\rho), \end{aligned}\right\} \quad \text{(A.3)}$$

so it follows immediately from (A.1–A.3) that

$$V_x - U_x = O(U_r). \quad \text{(A.4)}$$

From a consideration of the magnitude of the radial and axial mean velocities in the clean jet, (A.4) indicates that over most of the jet

$$V_x \simeq U_x \tag{A.5}$$

with a typical error of order 1%.

The corresponding magnitude analysis of terms in the radial momentum equation shows that $|(V_r - U_r)/U_r|$ may be of order unity. However, the time scale for changes in the mean velocity field is large compared with t_*, the response time of the particles, so we can assume that along with the axial components the radial components of mean velocity will be approximately equal,

$$V_r \simeq U_r. \tag{A.6}$$

UN MODELE DU JET TURBULENT DIPHASIQUE

Résumé—On présente un modèle du jet turbulent diphasique. On considère le cas où le fluide primaire contient une phase secondaire de particules rigides. La fraction massique de la phase secondaire est, la plupart du temps, de l'ordre de l'unité tandis que la fraction volumique est plutôt inférieure à l'unité. Un système d'équations aux dérivées partielles est développé pour les cas dans lesquels les vitesses moyennes des phases sont sensiblement égales. Une hypothèse de fermeture au premier ordre est utilisée et les équations sont résolues numériquement. On suppose pour la quantité de mouvement un transfert entre les phases et la réponse imparfaite des particules à la turbulence du fluide. Un accord satisfaisant est obtenu avec les données expérimentales publiées pour les valeurs calculées de la vitesse moyenne et du flux massique moyen des particules.

EIN MODELL FÜR DEN TURBULENTEN ZWEIPHASENSTRAHL

Zusammenfassung—Es wird ein Modell des turbulenten Zweiphasenstrahls vorgeschlagen. Es werden die Fälle betrachtet, bei denen die primäre flüssige Phase eine sekundäre Phase fester Partikel enthält. Der Massenanteil der sekundären Phase ist im Höchstfall von der Größenordnung 1, während ihr Volumenanteil viel kleiner als 1 ist. Es wird ein Satz von Modell-Differentialgleichungen für Fälle entwickelt, bei denen die mittleren Geschwindigkeiten der Phasen in etwa gleich sind. Ein Lösungsschema erster Ordnung wird für den achsensymmetrischen Strahl entwickelt und die sich ergebenden Gleichungen numerisch gelöst. Dieses Schema berücksichtigt den Impulsaustausch zwischen den Phasen und die unvollkommene Anpassung der Partikel an die turbulenten Flüssigkeitsbewegungen. Es wird eine befriedigende Übereinstimmung der berechneten Werte der mittleren Geschwindigkeit und des mittleren Massenstromes der Partikel mit veröffentlichten Meßwerten erzielt.

МОДЕЛЬ ДВУХФАЗНОЙ ТУРБУЛЕНТНОЙ СТРУИ

Аннотация — Представлена модель двухфазной турбулентной струи. Рассматриваются случаи, когда в основной жидкой фазе содержатся твердые частицы. Массовое содержание твердой фазы составляет величину порядка единицы, в то время как объёмное содержание значительно меньшую величину. Разработана система модельных дифференциальных уравнений для случаев, когда средние скорости фаз в основном одинаковы. Для расчёта осесимметричной струи предложена модель замыкания первого порядка. Полученные уравнения решены численно. Модель учитывает перенос импульса между фазами и частично влияние турбулентности жидкости на динамику частиц. Получено удовлетворительное соответствие с опубликованными экспериментальными данными для рассчитанных значений средней скорости и среднего массового потока частиц.