RESONANCE OF SHELF WAVES NEAR ISLANDS

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The linear, inviscid equations of motion of shallow-water waves in a rotating frame of reference are

$$\frac{\partial \underline{u}}{\partial t} + f\underline{k}\wedge\underline{u} = - g\underline{\nabla}\zeta, \tag{1}$$

where $\underline{u}$ is the horizontal velocity vector (averaged over the depth), ζ is the surface elevation, $\underline{k}$ is a unit vector vertically upwards, and $\frac{1}{2}f$ is the angular velocity about a vertical axis. The averaged equation of continuity is

$$\underline{\nabla}.(h\underline{u}) + \frac{\partial\zeta}{\partial t} = 0, \tag{2}$$

where h is the water depth.

Assuming dimensionless plane polar coordinates $r = r^1/b$, θ, let the variation with depth be of the form

$$h(r) = h_1 r^{\alpha} \tag{3}$$

where h_1, α are constants. Also assume that ζ is of the form

$$\zeta = Z(r)\exp[i(n\theta-\omega t)], \tag{4}$$

where n is an integer. The resulting ordinary differential equation for Z is

$$r^2\frac{d^2Z}{dr^2} + (1+\alpha)r\frac{dZ}{dr} - (\alpha\mu n + n^2 + \delta^2 r^{2-\alpha})Z = 0, \tag{5}$$

and the radial component of the velocity is given by

$$\omega(1-\mu^2)ru_r = - gi(r\frac{dZ}{dr} - \mu nZ), \tag{6}$$

where $\mu = f/\omega$, and $\delta^2 = \varepsilon(1-\mu^{-2})$. The number

$$\varepsilon = fb/(gh_1)^{\frac{1}{2}} \tag{7}$$

is the divergence parameter. This parameter is very small for oceanic scales, so that a theory in which $\varepsilon = 0$, and $R \to \infty$ is satisfactory for island oscillations, (Longuet-Higgins, [1]). However, in the laboratory ε is not small, and a non-divergent theory is not applicable.

In the case of uniform depth, $\alpha = 0$, when (5) reduces to the modified Bessel equation

$$r^2\frac{d^2Z}{dr^2} + r\frac{dZ}{dr} - (n^2+\delta^2)Z = 0 ,$$

with solutions of the form

$$Z = A_1K_n(\delta r) + A_2I_n(\delta r), \tag{8}$$

where A_1, A_2 are arbitrary constants, and K_n, I_n are the modified Bessel functions. If $\alpha = 2$, (5) admits solutions of the form

$$Z = A_+r^{p_+} + A_-r^{p_-} ,$$

where

$$p_{\pm} = -1 \pm \sqrt{(1 + 2\mu n + n^2)}.$$

If $\alpha \neq 2$, the transformation

$$\rho^2 = r^{2-\alpha} \tag{9}$$

reduces (5) to

$$(1-\tfrac{1}{2}\alpha)^2\rho^2 \frac{d^2Z}{d\rho^2} + (1-\tfrac{1}{2}\alpha^2)\rho \frac{dZ}{d\rho} - (\rho^2+\mu\alpha n+\delta^2\rho^2)Z = 0, \tag{10}$$

the solution of which is easily seen to be a linear combination of $\rho^{-\lambda}J_{\pm\sigma}[2i\delta\rho/2-\alpha)]$, where $\lambda = \alpha/(2-\alpha)$ and $\sigma^2(2-\alpha)^2 = 4n^2 + 4\mu\alpha n + \alpha^2$. Since σ takes imaginary values, it is more convenient, however, to express the solution of (10) in the form

$$Z(\rho) = A_3\, X_+\,(\delta\rho) + A_4\, X_-\,(\delta\rho), \tag{11}$$

where $X_{\pm}(x)$ is defined by the series

$$x^{\lambda}X_{\pm}(x) = x^{\pm\sigma} \cdot \sum_{j=0}^{\infty} c_j\,[x/(2-\alpha)]^{2j}, \tag{12}$$

and $c_0 = 1$, $c_{j-i}/c_j = j(j \pm \sigma)$.

In the laboratory experiments of Caldwell and Eide [2] a rotating cylindrical tank of radius R contains an island, so that the water is in the annulus $a < r < R$, with the boundary conditions

$$u_r = 0 \text{ at } \qquad r = a,\ R. \tag{13}$$

Around the island there is a shelf region with depth variation as in (3) for $a < r \leqslant 1$. For $1 \leqslant r < R$ the depth is the constant h_1. The conditions at $r = 1$ are that ζ and u_r are continuous.

The solutions in the respective annuli are given by (11) and (8). Using the boundary conditions at $r = a, 1, R$, we obtain the equations

$$\sum_{q=1}^{4} \alpha_{pq} A_q = 0, \tag{14}$$

where $\qquad \alpha_{11} = \alpha_{12} = \alpha_{43} = \alpha_{44} = 0$, and

$\alpha_{13} = R\,\delta\, K_n'(\delta R) - \mu n\, K_n(\delta R)$; $\alpha_{14} = R\delta I_n'(\delta R) - \mu n\, I_n(\delta r)$;

$\alpha_{21} = -X_+(\delta)$; $\alpha_{22} = -X_-(\delta)$; $\alpha_{23} = K_n(\delta)$; $\alpha_{24} = I_n(\delta)$;

$\alpha_{31} = -(1-\tfrac{1}{2}\alpha)X_+'(\delta)$; $\alpha_{32} = -(1-\tfrac{1}{2}\alpha)X_-'(\delta)$; $\alpha_{33} = K_n'(\delta)$; $\alpha_{34} = I_n'(\delta)$;

$$\alpha_{41} = a^{1-\frac{1}{2}\alpha}\delta(1-\tfrac{1}{2}\alpha)X_+'(\delta a) - \mu n\, X_+(\delta a);$$

$$\alpha_{42} = a^{1-\frac{1}{2}\alpha}\,(1-\tfrac{1}{2}\alpha)X_-'(\delta a) - \mu n\, X_-(\delta a).$$

The condition that (14) is consistent is that the determinant of the matrix α_{pq} vanishes, and given n, the zeros of the determinant give the resonant frequencies of the system. Note that all the resonances are present, not only of shelf waves, but also of the Kelvin/edge wave spectrum, using the terminology of Munk, Snodgrass and Wimbush [3].

The actual computations were carried out using a linear interpolation root finding method. The numerical values of the parameters were those used in the laboratory experiments, with $R = 1.68$, $b = 36.6$ cm., $\alpha = 1.21$, $f = 6.283$ rad./sec., and a^{-1} in the range 1.5 to 15. Using the notation (m, n) for the mode with m radial nodes and n azimuthal nodes, figures 1, 2 and 3 compare the calculated and experimental results.

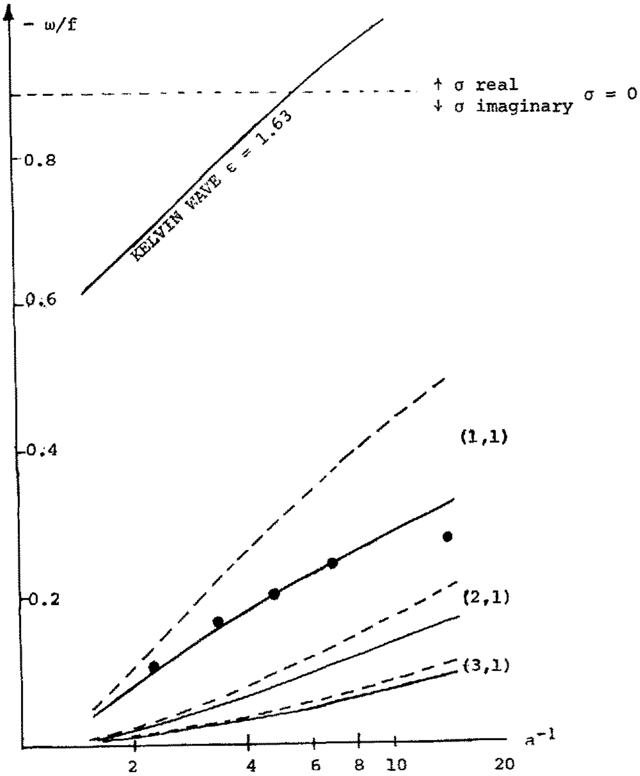

FIG. 1 Theoretical curves and experimental points relating resonance frequency $-\omega/f$ and a^{-1} for $n = 1$. The solid lines give the theoretical curves for the (1,1), (2,1) and (3,1) modes, for $\alpha = 1.21$, $h_1 = 20.3$ cm and $\varepsilon = 1.63$, and • denote the experimental points for the (1,1) mode. The broken lines ---- are the equivalent theoretical curves for $\varepsilon = 0$, and —— is the Kelvin-edge waves for $\varepsilon = 1.63$. Note that σ is imaginary for $-\omega/f < 0.88$, and real otherwise.

There is generally very good agreement in the range $1.5 < a^{-1} < 9$, but the results diverge for larger values of this parameter. There are several possible explanations. An obvious one is that the value $\alpha=1.21$ does not correspond to the experimental situation for large values of a^{-1}. In addition, the depth becomes very small for small values of r, so that friction and non-linear effects not considered here are likely to be significant. Nevertheless, the good agreement in the main range does explain the bulk of the experimental results.

Figure 4 illustrates the variation of the resonant frequency in the (1,1) mode with values of R. It is seen that in the actual laboratory experiments, the effect of the outer container wall is not very important, but that it is important for slightly smaller container radii.

Finally, as well as shelf waves, there are present Kelvin wave and edge wave modes. It was found that for $R=1.68$, $n=1$, there is a resonance corresponding to a Kelvin wave travelling along the outer boundary in the opposite sense to the shelf waves, with a frequency $\omega/f=0.214$. The frequency was quite insensitive to variation in a. An edge wave mode was also found, travelling in the same direction as the shelf waves, and this is indicated appropriately in Figure 1.

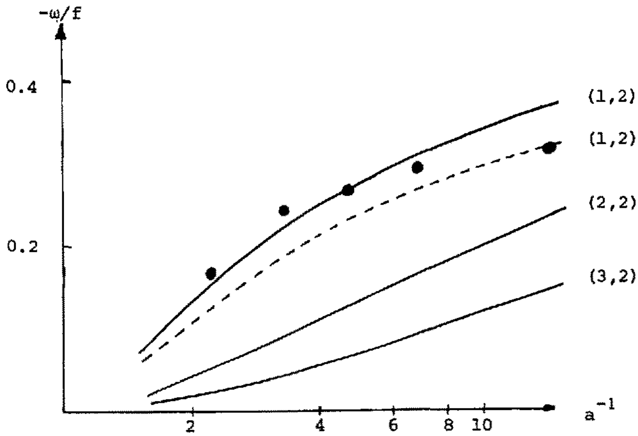

FIG. 2 Theoretical curves for the (1,2), (2,2) and (3,2) modes for α = 1,21, ε = 1.63 and experimental points for the (1,2) mode. The broken line ---- is the theoretical curve for the (1,2) mode with α = 1.0.

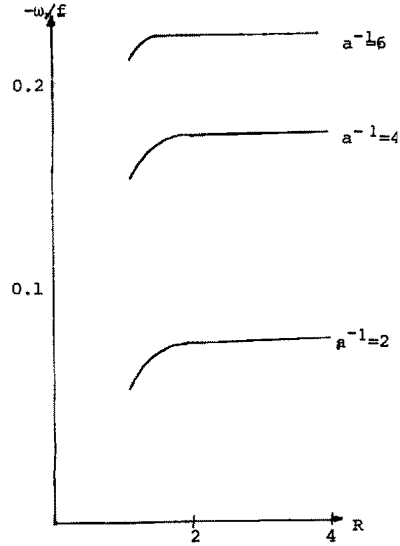

FIG. 4 Variation of resonant frequency - ω/f against R'/b for different values of a^{-1}.

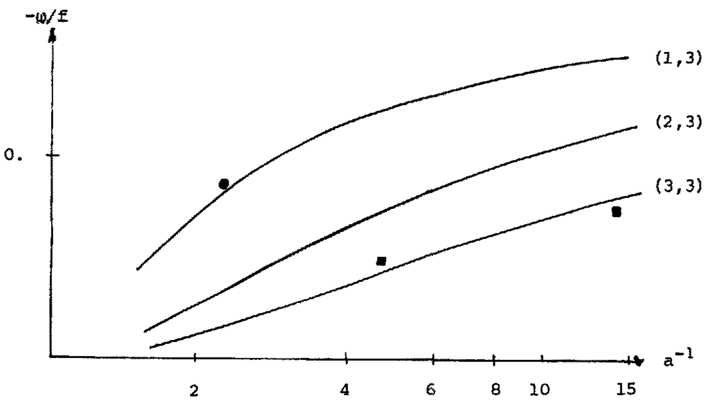

FIG. 3 Theoretical curves for the (1,3), (2,3) and (3,3) modes. The only experimental point for the (1,3) mode is given by •, and the experimental points are stated by Caldwell and Eide *to be for the (2,3) mode, but it appears from the theory that* they are actually for the (3,3) mode.

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REFERENCES

1. Longuet-Higgins, M.S., Steady currents induced by oscillations round islands, J.Fluid Mech., *42*, (1970), 701 - 720.
2. Caldwell, D.R. and Eide, S.A., Experiments on the resonance of long-period waves near islands, private communication, to be published.
3. Munk, W.H., Snodgrass, F. and Wimbush, M. Tides Offshore: Transition from California Coastal to Deep-Sea waters, Geophys. Fluid Dyn., *1*, (1970), 161 - 236.